

## ON HOPF ALGEBRAS OVER QUANTUM SUBGROUPS

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ABSTRACT. Using the standard filtration associated to a generalized lifting method, we determine all finite-dimensional Hopf algebras over an algebraically closed field of characteristic zero whose coradical generates a Hopf subalgebra isomorphic to the smallest non-pointed Hopf algebra  $\mathcal{K}$  of dimension 8 and the corresponding infinitesimal module is an indecomposable object in  ${}^{\mathcal{K}}\mathcal{YD}$ . As a byproduct we obtain new Hopf algebras of dimension 64.

## INTRODUCTION

Let  $\mathbb{k}$  be an algebraically closed field of characteristic zero. The question of classification of all Hopf algebras over  $\mathbb{k}$  of a given dimension up to isomorphism was posed by Kaplansky in 1975 [K]. Some progress has been made but, in general, it is a difficult question where there are no standard methods. One of the few general techniques is the so-called *Lifting Method* [AS2], under the assumption that the coradical is a subalgebra, *i.e.* the Hopf algebra has the Chevalley Property. More recently, Andruskiewitsch and Cuadra [AC] proposed to extend this technique by considering the subalgebra generated by the coradical and the related filtration. It turns out that this filtration is a Hopf algebra filtration, provided that the antipode is injective.

We describe the lifting method briefly. Let  $H$  be a Hopf algebra. Recall that the coradical filtration  $\{H_n\}_{n \geq 0}$  of  $H$  is defined recursively by

- $H_0$  is the coradical,
- $H_n = \bigwedge^{n+1} H_0 = \{h \in H : \Delta(h) \in H \otimes H_0 + H_n \otimes H\}$ .

This filtration corresponds to the filtration of  $H^*$  given by the powers of the Jacobson radical. It is always a coalgebra filtration and if  $H_0$  is a Hopf subalgebra, then it is also an algebra filtration; in particular, its associated graded object  $\text{gr } H = \bigoplus_{n \geq 0} H_n/H_{n-1}$  is a graded Hopf algebra, where  $H_{-1} = 0$ . Let  $\pi : \text{gr } H \rightarrow H_0$  be the homogeneous projection. It turns out that  $\text{gr } H \simeq R \# H_0$  as Hopf algebras, where  $R = (\text{gr } H)^{\text{co } \pi} = \{h \in H : (\text{id} \otimes \pi)\Delta(h) = h \otimes 1\}$  and  $\#$  stands for the Radford-Majid biproduct or *bosonization* of  $R$  with  $H_0$ .  $R$  is not a usual Hopf algebra, but a graded connected Hopf algebra in the category  ${}^{H_0}_{H_0}\mathcal{YD}$  of Yetter-Drinfeld modules over  $H_0$ . It contains the algebra generated by the elements of degree one, called the *Nichols algebra*  $\mathfrak{B}(V)$  of  $V$ ; here  $V = R^1$  is a braided vector space called the *infinitesimal braiding*.

Assume we have a fixed cosemisimple Hopf algebra  $A$ . The lifting method then consists of the description of all graded connected braided Hopf algebras  $R \in {}^A_A\mathcal{YD}$ , the determination of all possible deformations of the bosonization  $R \# A$  and the proof that all Hopf algebras  $H$  with  $H_0 = A$  satisfy that  $\text{gr } H \simeq R \# A$ . In general, each step of the method constitute a difficult problem to solve. Through the use of the lifting method the complete classification, with non-trivial examples, of finite-dimensional pointed Hopf algebras  $H$  with  $H_0$  a group algebra  $\mathbb{k}G$ , was obtained in the following cases

- $G$  a finite abelian group such that  $(|G|, 210) = 1$  [AS3],
- $G = \mathbb{S}_3, \mathbb{S}_4$  the symmetric groups in 3 and 4 letters [AHS], [GGI].
- $G = \mathbb{D}_{4t}$  the dihedral groups of order  $8t$ ,  $t \geq 3$  [FG].

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- $G$  any group admitting a principal realization over some affine racks, *e.g.*  $G \simeq \mathbb{Z}_5 \rtimes \mathbb{Z}_m$  [GIV].

Using different techniques, it was also shown that finite-dimensional pointed Hopf algebras over some finite simple non-abelian groups  $G$  are trivial, that is, isomorphic to group algebras:

- alternating groups  $\mathbb{A}_n$  with  $n \geq 5$  [AFGV1],
- most simple sporadic groups [AFGV2], [FV],
- infinite families of the projective special linear groups  $\mathbf{PSL}(n, q)$  and the special linear groups  $\mathbf{SL}(n, q)$  over finite fields  $\mathbb{F}_q$  [ACG], [FGV].

The lifting method is also effective to study finite-dimensional *copointed* Hopf algebras, that is, Hopf algebras such that its coradical is isomorphic to a function algebra over a finite group  $G$ . In this case, the classification is known for

- $G = \mathbb{S}_3$  [AV],
- $G$  any group admitting a principal realization over some affine racks [GIV].

The method also works for infinite-dimensional Hopf algebras of finite Gelfand-Kirillov dimension [AAH] or finite-dimensional Hopf algebras over finite fields.

The main idea in [AC] is to replace the coradical filtration by a more general but adequate filtration: the **standard filtration**  $\{H_{[n]}\}_{n \geq 0}$ , is defined recursively by

- $H_{[0]}$  to be the subalgebra generated by  $H_0$ , called the *Hopf coradical*,
- $H_{[n]} = \bigwedge^{n+1} H_{[0]}$ .

If the coradical  $H_0$  is a Hopf subalgebra, then  $H_{[0]} = H_0$  and the coradical filtration coincides with the standard one.

Let  $A$  be an arbitrary Hopf algebra. We will say that  $H$  is a *Hopf algebra over  $A$*  if  $H_{[0]} \simeq A$  as Hopf algebras.

Assume the antipode  $\mathcal{S}$  of  $H$  is injective, then by [AC, Lemma 1.1] it holds that  $H_{[0]}$  is a Hopf subalgebra of  $H$ ,  $H_n \subseteq H_{[n]}$  and  $\{H_{[n]}\}_{n \geq 0}$  is a Hopf algebra filtration of  $H$ . In particular, the graded algebra  $\text{gr } H = \bigoplus_{n \geq 0} H_{[n]}/H_{[n-1]}$  with  $H_{[-1]} = 0$  is a Hopf algebra associated with the standard filtration. If we denote as before,  $\pi : \text{gr } H \rightarrow H_{[0]}$  the homogeneous projection, it splits the inclusion of  $H_{[0]}$  in  $\text{gr } H$ , the *diagram*  $R = (\text{gr } H)^{\text{co } \pi}$  is a Hopf algebra in the category  ${}^{H_{[0]}}_{H_{[0]}}\mathcal{YD}$  of Yetter-Drinfel'd modules over  $H_{[0]}$  and  $\text{gr } H \simeq R \# H_{[0]}$  as Hopf algebras. It turns out that  $R$  is also graded and connected. We call again the linear space  $R^1$  corresponding to degree one the *infinitesimal braiding*. This is summarized in the following theorem.

**Theorem.** [AC, Theorem 1.3] *Any Hopf algebra with injective antipode is a deformation of the bosonization of a Hopf algebra generated by a cosemisimple coalgebra by a connected graded Hopf algebra in the category of Yetter-Drinfeld modules over the latter.*  $\square$

The procedure to describe explicitly any Hopf algebra as above defines a proposal for the classification of general Hopf algebras with injective antipode over a fixed Hopf subalgebra  $A$  which is generated by a cosemisimple coalgebra. The main steps are the following:

- determine all Yetter-Drinfeld modules  $V$  in  ${}^A_A\mathcal{YD}$  such that the Nichols algebra  $\mathfrak{B}(V)$  is finite dimensional,
- for such  $V$ , compute all Hopf algebras  $L$  such that  $\text{gr } L \simeq \mathfrak{B}(V) \# A$ . We call  $L$  a *lifting* of  $\mathfrak{B}(V)$  over  $A$ .
- Prove that any finite-dimensional Hopf algebra over  $A$  is generated by the first term of the standard filtration.

In this paper, we study this question in the case that  $A = \mathcal{K}$  is the smallest Hopf algebra generated by a simple coalgebra of dimension 4. It is an 8-dimensional Hopf algebra which is the dual of a pointed Hopf algebra. As algebra,  $\mathcal{K}$  is generated by the elements  $a, b, c, d$

satisfying the relations

$$\begin{aligned} ab &= \xi ba, & ac &= \xi ca, & 0 &= cb = bc, & cd &= \xi dc, & bd &= \xi db, \\ ad &= da, & ad &= 1, & 0 &= b^2 = c^2, & a^2c &= b, & a^4 &= 1. \end{aligned}$$

The coalgebra structure and its antipode are determined by

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, & \varepsilon(a) &= 1, & \varepsilon(b) &= 0, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, & \varepsilon(c) &= 1, & \varepsilon(d) &= 1, \\ \mathcal{S}(a) &= d, & \mathcal{S}(b) &= \xi b, & \mathcal{S}(c) &= -\xi c, & \mathcal{S}(d) &= a. \end{aligned}$$

See Section 2 for more details. Notice that, by a result of Stefan [S, Theorem 1.5],  $\mathcal{K}$  is a *quantum subgroup* of  $\mathbf{SL}_\xi(2)$  since it is a quotient of the quantum group  $\mathcal{O}_\xi(\mathbf{SL}_2)$ . For this reason, we call any Hopf algebra over  $\mathcal{K}$ , a Hopf algebra over a quantum subgroup.

In order to determine finite-dimensional Hopf algebras over  $\mathcal{K}$ , we first compute the Drinfeld double  $D := D(\mathcal{K}^{\text{cop}})$  of  $\mathcal{K}^{\text{cop}}$  and describe the simple and indecomposable left  $D$ -modules, and the projective covers of the simple modules. In fact, we prove in Theorem 2.9 that there are 16 simple left  $D$ -modules pairwise non-isomorphic. Four one-dimensional given by characters and 12 two-dimensional. The former correspond to characters on  $\mathbb{Z}_4$  and the latter are parametrized by the finite subset of  $\mathbb{Z}_4 \times \mathbb{Z}_4$  given by  $\Lambda = \{(i, j) \in \mathbb{Z}_4 \times \mathbb{Z}_4 \mid 2i \neq j\}$ . We compute the separation diagram of  $D$  and show that  $D$  is of tame representation type.

Using that the categories  ${}_D\mathcal{M}$  and  ${}_{\mathcal{K}}\mathcal{YD}$  are equivalent, we then translate the description above to simple and indecomposable modules in  ${}_{\mathcal{K}}\mathcal{YD}$  and prove in Theorem 4.5 that the Nichols algebra  $\mathfrak{B}(M)$  is infinite-dimensional for any finite-dimensional non-simple indecomposable module  $M \in {}_{\mathcal{K}}\mathcal{YD}$ . In particular, if  $\mathfrak{B}(V)$  is finite-dimensional, then  $V$  must be a semisimple module. Then, using the description of the braiding in  ${}_{\mathcal{K}}\mathcal{YD}$ , we obtain our first main result, see Section 3 for definitions.

**Theorem A.** *Let  $\mathfrak{B}(V)$  be a finite-dimensional Nichols algebra over an indecomposable object in  ${}_{\mathcal{K}}\mathcal{YD}$ . Then  $V$  is simple and isomorphic either to  $\mathbb{k}_\chi$ ,  $\mathbb{k}_{\chi^3}$ ,  $V_{2,1}$ ,  $V_{2,3}$ ,  $V_{3,1}$  or  $V_{3,3}$ .*

It turns out that  $\mathfrak{B}(\mathbb{k}_{\chi^\ell}) \simeq \bigwedge \mathbb{k}_{\chi^\ell}$  is an exterior algebra for  $\ell = 1, 3$  with  $\dim \mathfrak{B}(\mathbb{k}_{\chi^\ell}) = 2$  and  $\mathfrak{B}(V)$  is an 8-dimensional algebra for  $V = V_{2,1}$ ,  $V_{2,3}$ ,  $V_{3,1}$  and  $V_{3,3}$ . Since all objects in  ${}_{\mathcal{K}}\mathcal{YD}$  can be described as objects in the category of Yetter-Drinfeld modules over the pointed Hopf algebra  $\mathcal{K}^* = \mathcal{A}_4''$ , by [U] it follows that the associated braiding is triangular. These 8-dimensional examples are new examples of finite-dimensional Nichols algebras. They are isomorphic to quantum linear spaces as algebras, but not as coalgebras since the braiding differs; in our case, the braiding is not diagonal, see the Appendix.

As the study of Nichols algebras over semisimple modules is a hard problem that demands different techniques to be applied, we focus on the description of Hopf algebras over  $\mathcal{K}$  such that their infinitesimal braiding is simple. After proving in Theorem 5.1 that any such Hopf algebra is generated in degree one with respect to the standard filtration, we define two Hopf algebras  $\mathfrak{A}_{3,1}(\mu)$  and  $\mathfrak{A}_{3,3}(\mu)$  depending on a parameter  $\mu \in \mathbb{k}$  and prove our second main result, see Section 5 for definitions.

**Theorem B.** *Let  $H$  be a finite-dimensional Hopf algebra over  $\mathcal{K}$  such that its infinitesimal braiding is an indecomposable module  $V$  in  ${}_{\mathcal{K}}\mathcal{YD}$ . Then  $V$  is simple and  $H$  is isomorphic either to*

- (i)  $\bigwedge \mathbb{k}_{\chi^\ell} \# \mathcal{K}$  with  $\ell = 1, 3$ ;
- (ii)  $\mathfrak{B}(V_{2,1}) \# \mathcal{K}$ ;
- (iii)  $\mathfrak{B}(V_{2,3}) \# \mathcal{K}$ ;
- (iv)  $\mathfrak{A}_{3,1}(\mu)$  for some  $\mu \in \mathbb{k}$ ;
- (v)  $\mathfrak{A}_{3,3}(\mu)$  for some  $\mu \in \mathbb{k}$ .

The Hopf algebras  $\bigwedge \mathbb{k}_{\chi^\ell} \# \mathcal{K}$  with  $\ell = 1, 3$  have dimension 16 and are duals of pointed Hopf algebras. They have already appeared in [B]. The Hopf algebras  $\mathfrak{B}(V_{2,1}) \# \mathcal{K}$  and

$\mathfrak{B}(V_{2,3})\#\mathcal{K}$  are dual of pointed Hopf algebras of dimension 64. The Hopf algebras  $\mathfrak{A}_{3,1}(\mu)$  and  $\mathfrak{A}_{3,3}(\mu)$  are non-pointed with non-pointed duals. As far as the authors knowledge, they constitute new examples of Hopf algebras of dimension 64.

The paper is organized as follows. In Section 1 we recall some invariants associated to a Hopf algebra, define Yetter-Drinfeld modules, Nichols algebras and the Drinfeld double construction, and recall the relation between Hopf algebras with a projection and bosonizations. In Section 2 we describe the structure of  $\mathcal{K}$  and give the presentation of the double  $D = D(\mathcal{K}^{\text{cop}})$  by generators and relations. We also determine the simple left  $D$ -modules, their projective covers and some indecomposable left  $D$ -modules. We compute the Ext-Quiver of  $D$  and show that  $D$  is of tame representation type.

Then, using the equivalence  ${}_D\mathcal{M} \simeq {}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$ , we determine in Section 3 the corresponding objects of the latter and describe their braidings. The braided vector spaces corresponding to 2-dimensional simple modules are not diagonal. We give the proof of this fact in the Appendix.

In Section 4 we show that if  $\mathfrak{B}(V)$  is a finite-dimensional Nichols algebra in  ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$ , then  $V$  is necessarily a semisimple object and prove Theorem A by describing first the Nichols algebra of the simple modules.

Finally, in Section 5 we first show that a finite-dimensional Hopf algebra over  $\mathcal{K}$  whose infinitesimal braiding is a simple module in  ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$  is generated by the first term of the standard filtration, and then prove Theorem B.

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## 1. PRELIMINARIES

**1.1. Conventions.** We work over an algebraically closed field  $\mathbb{k}$  of characteristic zero. Our references for Hopf algebra theory are [M], [R] and [Sw].

If  $H$  is a Hopf algebra over  $\mathbb{k}$  then  $\Delta$ ,  $\varepsilon$  and  $\mathcal{S}$  denote respectively the comultiplication, the counit and the antipode. Comultiplication and coactions are written using the Sweedler notation with summation sign suppressed, *e.g.*,  $\Delta(h) = h_{(1)} \otimes h_{(2)}$  for  $h \in H$ . If  $H$  is a Hopf algebra in a braided monoidal category, we call it a *braided* Hopf algebra. Usual Hopf algebras are Hopf algebras in the category  $\text{Vect}_{\mathbb{k}}$  of  $\mathbb{k}$ -vector spaces. We denote by  ${}_H\mathcal{M}$  the category of finite-dimensional left  $H$ -modules.

The set  $G(H) = \{h \in H \setminus \{0\} : \Delta(h) = h \otimes h\}$  denotes the group of *group-like elements*. The *coradical*  $H_0$  of  $H$  is the sum of all simple subcoalgebras of  $H$ ; in particular,  $\mathbb{k}G(H) \subseteq H_0$ . If  $H$  has injective antipode, the algebra  $H_{[0]}$  generated by  $H_0$  is a Hopf subalgebra which is called the *Hopf coradical*. For  $h, g \in G(H)$ , the linear space of  $(h, g)$ -*primitives* is:

$$\mathcal{P}_{h,g}(H) := \{x \in H \mid \Delta(x) = x \otimes h + g \otimes x\}.$$

If  $g = 1 = h$ , the linear space  $\mathcal{P}(H) = \mathcal{P}_{1,1}(H)$  is called the set of primitive elements.

If  $M$  is a right  $H$ -comodule via  $\delta(m) = m_{(0)} \otimes m_{(1)} \in M \otimes H$  for all  $m \in M$ , then the space of *right coinvariants* is  $M^{\text{co}\delta} = \{x \in M \mid \delta(x) = x \otimes 1\}$ . In particular, if  $\pi : H \rightarrow L$  is a morphism of Hopf algebras, then  $H$  is a right  $L$ -comodule via  $(\text{id} \otimes \pi)\Delta$  and

$$H^{\text{co}\pi} := H^{\text{co}(\text{id} \otimes \pi)\Delta} = \{h \in H \mid (\text{id} \otimes \pi)\Delta(h) = h \otimes 1\}.$$

Left coinvariants, written  ${}^{\text{co}\pi}H$  are defined analogously.

**1.2. Yetter-Drinfeld modules and Nichols algebras.** Let  $H$  be a Hopf algebra. A left Yetter-Drinfeld module  $M$  over  $H$  is a left  $H$ -module  $(M, \cdot)$  and a left  $H$ -comodule  $(M, \delta)$  with  $\delta(m) = m_{(-1)} \otimes m_{(0)} \in H \otimes M$  for all  $m \in M$ , satisfying

$$\delta(h \cdot m) = h_{(1)} m_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)} \quad \forall m \in M, h \in H.$$

We denote by  ${}^H_H\mathcal{YD}$  the category of left Yetter-Drinfeld modules over  $H$ . It is a braided monoidal category: for  $M, N \in {}^H_H\mathcal{YD}$ , the braiding  $c_{M,N} : M \otimes N \rightarrow N \otimes M$  is given by

$$(1) \quad c_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)} \quad \forall m \in M, n \in N.$$

**Definition 1.1.** [AS2, Def. 2.1] Let  $H$  be a Hopf algebra and  $V \in {}^H_H\mathcal{YD}$ . A braided  $\mathbb{N}$ -graded Hopf algebra  $R = \bigoplus_{n \geq 0} R(n) \in {}^H_H\mathcal{YD}$  is called the Nichols algebra of  $V$  if

- (i)  $\mathbb{k} \simeq R(0)$ ,  $V \simeq R(1) \in {}^H_H\mathcal{YD}$ ,
- (ii)  $R(1) = \mathcal{P}(R) = \{r \in R \mid \Delta_R(r) = r \otimes 1 + 1 \otimes r\}$ .
- (iii)  $R$  is generated as an algebra by  $R(1)$ .

In this case,  $R$  is denoted by  $\mathfrak{B}(V) = \bigoplus_{n \geq 0} \mathfrak{B}^n(V)$ .

For any  $V \in {}^H_H\mathcal{YD}$  there is a Nichols algebra  $\mathfrak{B}(V)$  associated to it. It is the quotient of the tensor algebra  $T(V)$  by the largest homogeneous two-sided ideal  $I$  satisfying:

- $I$  is generated by homogeneous elements of degree  $\geq 2$ .
- $\Delta(I) \subseteq I \otimes T(V) + T(V) \otimes I$ , i. e., it is also a coideal.

In such a case,  $\mathfrak{B}(V) = T(V)/I$ . See [AS2, Section 2.1] for details.

**Remark 1.2.** An important observation is that the Nichols algebra  $\mathfrak{B}(V)$  is completely determined, as algebra and coalgebra, by the braiding. Let  $V$  be a vector space and  $c \in \text{End}(V \otimes V)$  be a solution of the braid equation  $(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)$ . Let  $AV$ ,  $AC$  be the tensor algebra and the cotensor algebra, respectively. Both are braided bialgebras and there exists a unique bialgebra map  $\mathbf{S} : AV \rightarrow CV$  such that  $\mathbf{S}|_V = \text{id}_V$ . The image  $\text{Im } \mathbf{S} \subseteq CV$  is a braided bialgebra called the quantum symmetric algebra. If the braiding is rigid, then  $\text{Im } \mathbf{S} = \mathfrak{B}(V)$  is a Nichols algebra; in such a case,  $\mathfrak{B}(V)$  is a braided Hopf algebra in a braided rigid category. See [AG] for details.

If  $W \subseteq V$  is a subspace such that  $c(W \otimes W) \subseteq W \otimes W$ , one may identify  $\mathfrak{B}(W)$  with a subalgebra of  $\mathfrak{B}(V)$ ; eventually belonging to different braided rigid categories. In particular, if  $\dim \mathfrak{B}(W) = \infty$ , then  $\dim \mathfrak{B}(V) = \infty$ . Thus, if  $V$  contains a non-zero element  $v$  such that  $c(v \otimes v) = v \otimes v$ , then  $\dim(V) = \infty$ .

**1.3. Bosonization and Hopf algebras with a projection.** Let  $H$  be a Hopf algebra and  $B$  a braided Hopf algebra in  ${}^H_H\mathcal{YD}$ . The procedure to obtain a usual Hopf algebra from  $B$  and  $H$  is called the Majid-Radford biproduct or *bosonization*, and it is usually denoted by  $B \# H$ . As vector spaces  $B \# H = B \otimes H$ , and the multiplication and comultiplication are given by the smash-product and smash-coproduct, respectively. That is, for all  $b, c \in B$  and  $g, h \in H$ , we have

$$(b \# g)(c \# h) = b(g_{(1)} \cdot c) \# g_{(2)} h, \\ \Delta(b \# g) = b^{(1)} \# (b^{(2)})_{(-1)} g_{(1)} \otimes (b^{(2)})_{(0)} \# g_{(2)},$$

where  $\Delta_B(b) = b^{(1)} \otimes b^{(2)}$  denotes the comultiplication in  $B \in {}^H_H\mathcal{YD}$ . If  $b \in B$  and  $h \in H$ , then we identify  $b = b \# 1$  and  $h = 1 \# h$ ; in particular we have  $bh = b \# h$  and  $hb = h_{(1)} \cdot b \# h_{(2)}$ . Clearly, the map  $\iota : H \rightarrow B \# H$  given by  $\iota(h) = 1 \# h$  for all  $h \in H$  is an injective Hopf algebra map, and the map  $\pi : B \# H \rightarrow H$  given by  $\pi(b \# h) = \varepsilon_B(b)h$  for all  $b \in B, h \in H$  is a surjective Hopf algebra map such that  $\pi \circ \iota = \text{id}_H$ . Moreover, it holds that  $B = (B \# H)^{\text{co } \pi}$ .

Conversely, let  $A$  be a Hopf algebra with bijective antipode and  $\pi : A \rightarrow H$  a Hopf algebra epimorphism admitting a Hopf algebra section  $\iota : H \rightarrow A$  such that  $\pi \circ \iota = \text{id}_H$ . Then  $B = A^{\text{co } \pi}$  is a braided Hopf algebra in  ${}^H_H\mathcal{YD}$  and  $A \simeq B \# H$  as Hopf algebras.



**1.4. The Drinfeld double.** We briefly describe the structure of the Drinfeld double of a finite-dimensional Hopf algebra.

**Definition 1.3.** [M, Def. 10.3.5] *Let  $H$  be a finite dimensional Hopf algebra. The Drinfeld double is the Hopf algebra  $D(H)$ , where  $D(H) = (H^*)^{\text{cop}} \bowtie H = H^* \otimes H$ , as vector spaces, the product and the unit are given by*

$$(f \bowtie h)(g \bowtie k) = f(h_1 \rightharpoonup g_2) \bowtie (h_2 \leftarrow g_1)k, \quad 1_{D(K)} = \varepsilon \bowtie 1,$$

where  $h \rightharpoonup f = \langle f_3(\mathcal{S}^*)^{-1}(f_1), h \rangle f_2$  and  $h \leftarrow f = \langle f, \mathcal{S}^{-1}(h_3)h_1 \rangle h_2$ , and the coproduct and counity are

$$\Delta(f \bowtie h) = f_2 \bowtie h_1 \otimes f_1 \bowtie h_2 \quad \varepsilon(f \bowtie h) = f(1)\varepsilon(h).$$

The following result establishes a categorical equivalence between  ${}^H_H\mathcal{YD}$  and  ${}_{D(H^{\text{cop}})}\mathcal{M}$ . This proposition will be central in Section 3, since we will study the simple and indecomposable left  $D(K^{\text{cop}})$ -modules first and then translate the information to  ${}^K_K\mathcal{YD}$ .

**Proposition 1.4.** [M, Prop. 10.6.16] *Let  $H$  be a finite-dimensional Hopf algebra. Then the Yetter-Drinfeld category  ${}^H_H\mathcal{YD}$  can be identified with the category  ${}_{D(H^{\text{cop}})}\mathcal{M}$  of left modules over the Drinfeld double  $D(H^{\text{cop}})$ .  $\square$*

## 2. THE QUANTUM SUBGROUP $\mathcal{K}$ AND ITS DRINFELD DOUBLE $D(K^{\text{cop}})$

In this section we describe the structure of the quantum subgroup  $\mathcal{K}$  and present the Drinfeld double  $D = D(K^{\text{cop}})$  by generators and relations. We also determine the simple left  $D$ -modules, their projective covers and some indecomposable left  $D$ -modules. We parametrize these objects by tuples of the characters of the commutative subalgebra generated by  $a, d$  and  $g$ . With this information we compute the Ext-Quiver of  $D$  and conclude that  $D$  is of tame representation type.

Throughout the paper, we fix  $\xi$  a primitive 4-th root of 1. All pointed nonsemisimple Hopf algebras of dimension 8 were determined by [S]. Except for one case, given by

$$\mathcal{A}_4'' := \mathbb{k}\langle g, x \mid g^4 - 1 = x^2 - g^2 + 1 = gx + xg = 0 \rangle,$$

with  $\Delta(g) = g \otimes g$  and  $\Delta(x) = x \otimes g + 1 \otimes x$ , these pointed Hopf algebras have pointed duals. Moreover, it holds that  $\mathcal{K} = (\mathcal{A}_4'')^*$ , see [GV]. Up to isomorphism,  $\mathcal{K}$  is the only Hopf algebra of dimension 8 which is neither semisimple nor pointed nor has the Chevalley property. The next proposition gives us a presentation of the Hopf algebra  $\mathcal{K}$  and some useful relations that will be used in the sequel. The proof follows from [GV, Lemma 3.3].

**Proposition 2.1.** (i)  $\mathcal{K}$  is generated as an algebra by the elements  $a, b, c, d$  satisfying

$$\begin{aligned} ab &= \xi ba, & ac &= \xi ca, & 0 &= cb = bc, & cd &= \xi dc, & bd &= \xi db, \\ ad &= da, & ad &= 1, & 0 &= b^2 = c^2, & a^2c &= b, & a^4 &= 1. \end{aligned}$$

(ii) A linear basis of  $\mathcal{K}$  is given by  $\{1, a, b, c, d, a^2, ab, ac\}$ .

(iii) The coalgebra structure is given by

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, & \Delta(c) &= c \otimes a + d \otimes c, \\ \Delta(d) &= c \otimes b + d \otimes d, & \Delta(ab) &= ab \otimes 1 + a^2 \otimes ab, & \Delta(ac) &= ac \otimes a^2 + 1 \otimes ac, \\ \Delta(a^2) &= a^2 \otimes a^2, & \varepsilon(a) &= \varepsilon(d) = 1, & \varepsilon(b) &= \varepsilon(c) = 0. \end{aligned}$$

(iv) The antipode is given by

$$\begin{aligned} \mathcal{S}(a) &= d, & \mathcal{S}(b) &= \xi b, & \mathcal{S}(c) &= -\xi c, & \mathcal{S}(d) &= a, \\ \mathcal{S}(a^2) &= a^2, & \mathcal{S}(ab) &= -ac, & \mathcal{S}(ac) &= ab. \end{aligned}$$

(v) The multiplication table is

	1	a	b	c	d	a <sup>2</sup>	ab	ac
1	1	a	b	c	d	a <sup>2</sup>	ab	ac
a	a	a <sup>2</sup>	ab	ac	1	d	c	b
b	b	-ξab	0	0	ξac	-c	0	0
c	c	-ξac	0	0	ξab	-b	0	0
d	d	1	ac	ab	a <sup>2</sup>	a	b	c
a <sup>2</sup>	a <sup>2</sup>	d	c	b	a	1	ac	ab
ab	ab	-ξc	0	0	ξb	-ac	0	0
ac	ac	-ξb	0	0	ξc	-ab	0	0

(vi)  $\mathcal{K} \simeq H_4 \oplus \mathcal{M}^*(2, \mathbb{k})$  as coalgebras.

□

**Remarks 2.2.** (a) Denote by  $\{1^*, a^*, b^*, c^*, d^*, (a^2)^*, (ab)^*, (ac)^*\}$  the basis of  $\mathcal{K}^*$  dual to  $\{1, a, b, c, d, a^2, ab, ac\}$ . Using the multiplication table in Proposition 2.1 (v), it follows that

$$\begin{aligned}
\Delta(1^*) &= 1^* \otimes 1^* + a^* \otimes d^* + d^* \otimes a^* + (a^2)^* \otimes (a^2)^*, \\
\Delta(a^*) &= 1^* \otimes a^* + a^* \otimes 1^* + (a^2)^* \otimes d^* + d^* \otimes (a^2)^*, \\
\Delta(d^*) &= 1^* \otimes d^* + d^* \otimes 1^* + (a^2)^* \otimes a^* + a^* \otimes (a^2)^*, \\
\Delta((a^2)^*) &= 1^* \otimes (a^2)^* + (a^2)^* \otimes 1^* + a^* \otimes a^* + d^* \otimes d^*, \\
\Delta(b^*) &= 1^* \otimes b^* + b^* \otimes 1^* + a^* \otimes (ac)^* - \xi(ac)^* \otimes a^* + \\
&\quad + (a^2)^* \otimes c^* - c^* \otimes (a^2)^* + \xi(ab)^* \otimes d^* + d^* \otimes (ab)^*, \\
\Delta(c^*) &= 1^* \otimes c^* + c^* \otimes 1^* - \xi(ab)^* \otimes a^* + a^* \otimes (ab)^* + \\
&\quad + (a^2)^* \otimes b^* - b^* \otimes (a^2)^* + \xi(ac)^* \otimes d^* + d^* \otimes (ac)^*, \\
\Delta((ab)^*) &= 1^* \otimes (ab)^* + (ab)^* \otimes 1^* - \xi b^* \otimes a^* + a^* \otimes b^* + \\
&\quad + d^* \otimes c^* + \xi c^* \otimes d^* - (ac)^* \otimes (a^2)^* + (a^2)^* \otimes (ac)^*, \\
\Delta((ac)^*) &= 1^* \otimes (ac)^* + (ac)^* \otimes 1^* - \xi c^* \otimes a^* + a^* \otimes c^* + \\
&\quad + d^* \otimes b^* + \xi b^* \otimes d^* - (ab)^* \otimes (a^2)^* + (a^2)^* \otimes (ab)^*.
\end{aligned}$$

(b) Let  $\alpha \in G(\mathcal{K}^*) = \text{Alg}(\mathcal{K}, \mathbb{k})$ . As  $a^4 = 1$  and  $ad = 1$ ,  $\alpha(a)$  is a 4-th root of unity and  $\alpha(d) = \alpha(a)^{-1}$ . Further, since  $b^2 = 0 = c^2$ , we have that  $\alpha(b) = 0 = \alpha(c)$ . Thus

$$G(\mathcal{K}^*) = \{\alpha_j = 1^* + \xi^{-j}a^* + \xi^j d^* + (-1)^j (a^2)^* : 0 \leq j \leq 3\}.$$

Note that  $\alpha_0 = \varepsilon$  and  $\alpha_1^j = \alpha_j$ . In particular,  $G(\mathcal{K}^*) \simeq \mathbb{Z}/4\mathbb{Z}$  and  $\alpha_\xi, \alpha_{-\xi}$  are generators.

(c) The multiplication table is

	1*	a*	b*	c*	d*	(a <sup>2</sup> )*	(ab)*	(ac)*
1*	1*	0	0	0	0	0	0	(ac)*
a*	0	a*	b*	0	0	0	0	0
b*	0	0	0	a*	b*	0	0	0
c*	0	c*	d*	0	0	0	0	0
d*	0	0	0	c*	d*	0	0	0
(a <sup>2</sup> )*	0	0	0	0	0	(a <sup>2</sup> )*	(ab)*	0
(ab)*	(ab)*	0	0	0	0	0	0	0
(ac)*	0	0	0	0	0	(ac)*	0	0

In order to compute the Drinfeld double  $D(\mathcal{K}^{\text{cop}})$  of  $\mathcal{K}^{\text{cop}}$  we need to describe the isomorphism  $\mathcal{K}^* \simeq \mathcal{A}_4''$  explicitly.

**Lemma 2.3.** The algebra map  $\varphi : \mathcal{A}_4'' \rightarrow \mathcal{K}^*$  given by

$$\varphi(g) = \alpha_1 = 1^* - \xi a^* + \xi d^* - (a^2)^* \quad \text{and} \quad \varphi(x) = \sqrt{2}\xi(b^* + c^* + (ab)^* + (ac)^*),$$

is a Hopf algebra isomorphism.

*Proof.* A direct computation shows that  $\varphi$  is a coalgebra map. Hence, the image of  $\varphi$  is a Hopf subalgebra of  $\mathcal{K}^*$  of dimension bigger than 4, since it contains the group algebra  $\mathbb{k}G(\mathcal{K}^*)$  and the image of the skew-primitive element  $x$ . Thus, by the Nichols-Zoeller theorem it follows that  $\varphi$  is surjective and whence an isomorphism.  $\square$

**Remark 2.4.** Let  $\{g^j, xg^j\}_{0 \leq j \leq 3}$  be a linear basis of  $\mathcal{A}_4''$ . By Remark 2.2 (c) and Lemma 2.3, it follows that

$$\begin{aligned} \varphi(g^j) &= \alpha_j = 1^* + \xi^{-j}a^* + \xi^j d^* + (-1)^j(a^2)^* & \text{for all } 0 \leq j \leq 3, \\ \varphi(xg^j) &= \sqrt{2}\xi(\xi^j b^* + \xi^{-j}c^* + (ab)^* + (-1)^j(ac)^*) & \text{for all } 0 \leq j \leq 3. \end{aligned}$$

**2.1. Description of  $D(\mathcal{K}^{\text{cop}})$ .** Now we describe the Drinfeld double  $D(\mathcal{K}^{\text{cop}})$  of  $\mathcal{K}^{\text{cop}}$ . To make the notation lighter, from now on we denote  $D = D(\mathcal{K}^{\text{cop}})$ .

**Proposition 2.5.**  $D(\mathcal{K}^{\text{cop}})$  is the  $\mathbb{k}$ -algebra generated by the elements  $a, b, c, d, x, g$  such that  $a, b, c, d$  satisfy the relations of  $\mathcal{K}^{\text{cop}}$ ,  $x, g$  satisfy the relations of  $(\mathcal{A}_4'')^{\text{op cop}}$  and

$$\begin{aligned} ax + \xi xa &= \sqrt{2}\xi(b + gc), & bx - \xi xb &= \sqrt{2}\xi(a - gd), \\ ag &= ga, & bg &= -gb, \\ cg &= -gc, & dg &= gd, \\ cx + \xi xc &= \sqrt{2}\xi(d - ga), & dx - \xi xd &= \sqrt{2}\xi(c + gb). \end{aligned}$$

*Proof.* Since  $(f \bowtie 1)(g \bowtie k) = fg \bowtie k$  and  $(f \bowtie h)(\varepsilon \bowtie k) = f \bowtie hk$  for all  $f, g \in (\mathcal{A}_4'')^{\text{op cop}}$  and  $h, k \in \mathcal{K}^{\text{cop}}$ , it is enough to describe the relations derived from products of the form  $(1_{\mathcal{A}_4''} \bowtie h)(y \bowtie 1_{\mathcal{K}})$ , where  $h \in \mathcal{K}^{\text{cop}}$  and  $y \in (\mathcal{A}_4'')^{\text{op cop}}$  are algebra generators.

Assume first that  $y = g$ . Since  $h \rightarrow g = \langle \mathcal{S}_{\text{cop}}^{-1}(g) \cdot_{\text{op}} g, h \rangle g = \langle g\mathcal{S}(g), h \rangle g = \langle 1, h \rangle g = \varepsilon(h)g$ , for all  $h \in \mathcal{K}^{\text{cop}}$ , it follows that

$$\begin{aligned} (1_{\mathcal{A}_4''} \bowtie \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix})(g \bowtie 1_{\mathcal{K}}) &= \begin{pmatrix} a_{(1)\text{cop}} \rightarrow g_{(2)} \bowtie a_{(2)\text{cop}} \leftarrow g_{(1)} \\ b_{(1)\text{cop}} \rightarrow g_{(2)} \bowtie b_{(2)\text{cop}} \leftarrow g_{(1)} \\ c_{(1)\text{cop}} \rightarrow g_{(2)} \bowtie c_{(2)\text{cop}} \leftarrow g_{(1)} \\ d_{(1)\text{cop}} \rightarrow g_{(2)} \bowtie d_{(2)\text{cop}} \leftarrow g_{(1)} \end{pmatrix} \\ &= \begin{pmatrix} a \rightarrow g \bowtie a \leftarrow g + c \rightarrow g \bowtie b \leftarrow g \\ b \rightarrow g \bowtie a \leftarrow g + d \rightarrow g \bowtie b \leftarrow g \\ a \rightarrow g \bowtie c \leftarrow g + c \rightarrow g \bowtie d \leftarrow g \\ b \rightarrow g \bowtie c \leftarrow g + d \rightarrow g \bowtie d \leftarrow g \end{pmatrix} = g \bowtie \begin{pmatrix} a \leftarrow g \\ b \leftarrow g \\ c \leftarrow g \\ d \leftarrow g \end{pmatrix} \\ &= g \bowtie \begin{pmatrix} \langle g, \mathcal{S}_{\text{cop}}^{-1}(a_{(3)\text{cop}})a_{(1)\text{cop}} \rangle a_{(2)\text{cop}} \\ \langle g, \mathcal{S}_{\text{cop}}^{-1}(b_{(3)\text{cop}})b_{(1)\text{cop}} \rangle b_{(2)\text{cop}} \\ \langle g, \mathcal{S}_{\text{cop}}^{-1}(c_{(3)\text{cop}})c_{(1)\text{cop}} \rangle c_{(2)\text{cop}} \\ \langle g, \mathcal{S}_{\text{cop}}^{-1}(d_{(3)\text{cop}})d_{(1)\text{cop}} \rangle d_{(2)\text{cop}} \end{pmatrix} = g \bowtie \begin{pmatrix} \langle g, \mathcal{S}(a_{(1)})a_{(3)} \rangle a_{(2)} \\ \langle g, \mathcal{S}(b_{(1)})b_{(3)} \rangle b_{(2)} \\ \langle g, \mathcal{S}(c_{(1)})c_{(3)} \rangle c_{(2)} \\ \langle g, \mathcal{S}(d_{(1)})d_{(3)} \rangle d_{(2)} \end{pmatrix} \\ &= g \bowtie \begin{pmatrix} \langle g, \mathcal{S}(a)a \rangle a + \langle g, \mathcal{S}(b)a \rangle c + \langle g, \mathcal{S}(a)c \rangle b + \langle g, \mathcal{S}(b)c \rangle d \\ \langle g, \mathcal{S}(a)b \rangle a + \langle g, \mathcal{S}(b)b \rangle c + \langle g, \mathcal{S}(a)d \rangle b + \langle g, \mathcal{S}(b)d \rangle d \\ \langle g, \mathcal{S}(c)a \rangle a + \langle g, \mathcal{S}(d)a \rangle c + \langle g, \mathcal{S}(c)c \rangle b + \langle g, \mathcal{S}(d)c \rangle d \\ \langle g, \mathcal{S}(c)b \rangle a + \langle g, \mathcal{S}(d)b \rangle c + \langle g, \mathcal{S}(c)d \rangle b + \langle g, \mathcal{S}(d)d \rangle d \end{pmatrix} \\ &= g \bowtie \begin{pmatrix} \langle g, da \rangle a + \langle g, \xi ba \rangle c + \langle g, dc \rangle b + \langle g, \xi bc \rangle d \\ \langle g, db \rangle a + \langle g, \xi bb \rangle c + \langle g, dd \rangle b + \langle g, \xi bd \rangle d \\ \langle g, -\xi ca \rangle a + \langle g, aa \rangle c + \langle g, -\xi cc \rangle b + \langle g, ac \rangle d \\ \langle g, -\xi cb \rangle a + \langle g, ab \rangle c + \langle g, -\xi cd \rangle b + \langle g, ad \rangle d \end{pmatrix} \\ &= g \bowtie \begin{pmatrix} \langle g, 1 \rangle a + \langle g, ab \rangle c + \langle g, ab \rangle b + \langle g, 0 \rangle d \\ \langle g, ac \rangle a + \langle g, 0 \rangle c + \langle g, a^2 \rangle b + \langle g, -ac \rangle d \\ \langle g, -ac \rangle a + \langle g, a^2 \rangle c + \langle g, 0 \rangle b + \langle g, ac \rangle d \\ \langle g, 0 \rangle a + \langle g, ab \rangle c + \langle g, ab \rangle b + \langle g, 1 \rangle d \end{pmatrix} = g \bowtie \begin{pmatrix} a \\ -b \\ -c \\ d \end{pmatrix}, \end{aligned}$$

which implies the relations  $ag = ga, bg = -gb, cg = -gc$  and  $dg = gd$ .



Assume now that  $y = x$ . Using the computations above, we have that

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \leftarrow x = \begin{pmatrix} \langle x, 1 \rangle a + \langle x, ab \rangle c + \langle x, ab \rangle b + \langle x, 0 \rangle d \\ \langle x, ac \rangle a + \langle x, 0 \rangle c + \langle x, a^2 \rangle b + \langle x, -ac \rangle d \\ \langle x, -ac \rangle a + \langle x, a^2 \rangle c + \langle x, 0 \rangle b + \langle x, ac \rangle d \\ \langle x, 0 \rangle a + \langle x, ab \rangle c + \langle x, ab \rangle b + \langle x, 1 \rangle d \end{pmatrix} = \sqrt{2}\xi \begin{pmatrix} b+c \\ a-d \\ d-a \\ b+c \end{pmatrix},$$

and

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rightarrow x &= \langle \mathcal{S}(x_{(1)})x_{(3)}, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle x_{(2)} \\ &= \langle \mathcal{S}(x)g, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle g + \langle \mathcal{S}(1)g, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle x + \langle \mathcal{S}(1)x, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle 1 \\ &= \langle -xg^3g, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle g + \langle g, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle x + \langle x, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle 1 \\ &= \langle x, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle (1-g) + \langle g, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rangle x = \begin{pmatrix} \xi x \\ \sqrt{2}\xi(1-g) \\ \sqrt{2}\xi(1-g) \\ -\xi x \end{pmatrix}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} (1_{\mathcal{A}_4''} \bowtie \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix})(x \bowtie 1_{\mathcal{K}}) &= \begin{pmatrix} a_{(2)} \rightarrow g \bowtie a_{(1)} \leftarrow x + a_{(2)} \rightarrow x \bowtie a_{(1)} \leftarrow 1 \\ b_{(2)} \rightarrow g \bowtie b_{(1)} \leftarrow x + b_{(2)} \rightarrow x \bowtie b_{(1)} \leftarrow 1 \\ c_{(2)} \rightarrow g \bowtie c_{(1)} \leftarrow x + c_{(2)} \rightarrow x \bowtie c_{(1)} \leftarrow 1 \\ d_{(2)} \rightarrow g \bowtie d_{(1)} \leftarrow x + d_{(2)} \rightarrow x \bowtie d_{(1)} \leftarrow 1 \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon(a_{(2)})g \bowtie a_{(1)} \leftarrow x + a_{(2)} \rightarrow x \bowtie a_{(1)} \\ \varepsilon(b_{(2)})g \bowtie b_{(1)} \leftarrow x + b_{(2)} \rightarrow x \bowtie b_{(1)} \\ \varepsilon(c_{(2)})g \bowtie c_{(1)} \leftarrow x + c_{(2)} \rightarrow x \bowtie c_{(1)} \\ \varepsilon(d_{(2)})g \bowtie d_{(1)} \leftarrow x + d_{(2)} \rightarrow x \bowtie d_{(1)} \end{pmatrix} \\ &= \begin{pmatrix} g \bowtie a \leftarrow x + a \rightarrow x \bowtie a + c \rightarrow x \bowtie b \\ g \bowtie b \leftarrow x + b \rightarrow x \bowtie a + d \rightarrow x \bowtie b \\ g \bowtie c \leftarrow x + a \rightarrow x \bowtie c + c \rightarrow x \bowtie d \\ g \bowtie d \leftarrow x + b \rightarrow x \bowtie c + d \rightarrow x \bowtie d \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2}\xi g \bowtie (b+c) + \xi x \bowtie a + \sqrt{2}\xi(1-g) \bowtie b \\ \sqrt{2}\xi g \bowtie (a-d) + \sqrt{2}\xi(1-g) \bowtie a + -\xi x \bowtie b \\ \sqrt{2}\xi g \bowtie (d-a) + \xi x \bowtie c + \sqrt{2}\xi(1-g) \bowtie d \\ \sqrt{2}\xi g \bowtie (b+c) + \sqrt{2}\xi(1-g) \bowtie c + -\xi x \bowtie d \end{pmatrix} \\ &= \begin{pmatrix} -\xi x \bowtie a + \sqrt{2}\xi g \bowtie c + \sqrt{2}\xi \bowtie b \\ \xi x \bowtie b - \sqrt{2}\xi g \bowtie d + \sqrt{2}\xi \bowtie a \\ -\xi x \bowtie c - \sqrt{2}\xi g \bowtie a + \sqrt{2}\xi \bowtie d \\ \xi x \bowtie d + \sqrt{2}\xi g \bowtie b + \sqrt{2}\xi \bowtie c \end{pmatrix}, \end{aligned}$$

which gives us the other four relations of  $D(\mathcal{K}^{\text{cop}})$ .  $\square$

**2.2. Simple left  $D$ -modules.** We begin by describing the one-dimensional  $D$ -modules.

**Lemma 2.6.** *There are four non-isomorphic one-dimensional left  $D$ -modules given by the characters  $\chi^j$ ,  $0 \leq j \leq 3$  where*

$$\chi^j(a) = \xi^j, \chi^j(b) = 0, \chi^j(c) = 0, \chi^j(d) = \xi^{-j}, \chi^j(x) = 0, \chi^j(g) = (-1)^j.$$

Moreover, any one-dimensional  $D$ -module is isomorphic to  $\mathbb{k}_{\chi^j}$  for some  $0 \leq j \leq 3$ .

*Proof.* Let  $\lambda : D \rightarrow \mathbb{k}$  be a character and write

$$\lambda(a) = \lambda_1, \lambda(b) = \lambda_2, \lambda(c) = \lambda_3, \lambda(d) = \lambda_4, \lambda(x) = \lambda_5, \lambda(g) = \lambda_6.$$

From the relations  $a^4 = 1 = g^4$ , it follows that  $\lambda(a)$  and  $\lambda(g)$  are 4-th roots of unity. Since  $ad = 1$ , we have that  $\lambda(d) = \lambda(a)^{-1}$ . As  $b^2 = c^2 = gx + xg = 0$ , we have that  $\lambda(b) = \lambda(c) = \lambda(x) = 0$  and then from  $g^2 = 1 + x^2$ , it follows that  $\lambda(g)^2 = 1$ . Since  $bx + \xi xb = \sqrt{2}\xi(d - ga)$ , it follows that  $\sqrt{2}\xi(\lambda(a)^{-1} - \lambda(g)\lambda(a)) = 0$  which means that  $\lambda(g) = \lambda(a)^{-2} = \lambda(a)^2$ . Hence,  $\lambda$  is determined by its value in  $a$  and consequently must equal  $\chi^j$  for some  $0 \leq j \leq 3$ . It is immediate that these modules are pairwise non-isomorphic.  $\square$

Note that for all  $0 \leq j \leq 3$ ,  $\chi^j$  coincides with the  $j$ th-power of the character  $\chi = \chi^1$  in the convolution algebra  $\text{Hom}_{\mathbb{k}}(D, \mathbb{k})$ . Thus, we will denote  $\chi^j$  with  $j \in \mathbb{Z}_4$ .

Next we describe the simple  $D$ -modules of dimension two. For this, consider the finite subset of  $\mathbb{Z}_4 \times \mathbb{Z}_4$  given by

$$\Lambda = \{(i, j) \in \mathbb{Z}_4 \times \mathbb{Z}_4 \mid 2i \neq j\}.$$

Clearly,  $|\Lambda| = 12$ .

**Lemma 2.7.** *For any pair  $(i, j) \in \Lambda$ , there exists a simple left  $D$ -module  $V_{i,j}$  of dimension 2. If we denote  $\lambda_1 = \xi^i$  and  $\lambda_2 = \xi^j$ , the action on a fixed basis is given by*

$$\begin{aligned} \rho_{i,j}(a) &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\xi\lambda_1 \end{pmatrix}, \quad \rho_{i,j}(b) = \begin{pmatrix} 0 & \lambda_1^2 \\ 0 & 0 \end{pmatrix}, \quad \rho_{i,j}(c) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \rho_{i,j}(d) &= \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \xi\lambda_1^{-1} \end{pmatrix}, \quad \rho_{i,j}(g) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \\ \rho_{i,j}(x) &= \begin{pmatrix} 0 & \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_1^3\lambda_2) \\ \sqrt{2}\xi(\lambda_1^3 - \lambda_1\lambda_2) & 0 \end{pmatrix}, \end{aligned}$$

Moreover, if  $V$  is a simple  $D$ -module of dimension 2, then  $V$  is isomorphic to  $V_{i,j}$  for some  $(i, j) \in \Lambda$  and  $V_{i,j} \simeq V_{k,\ell}$  if and only if  $(i, j) = (k, \ell)$ .

*Proof.* Let

$$\begin{aligned} \rho(a) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \\ \rho(d) &= \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \quad \rho(x) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad \rho(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \end{aligned}$$

be a two-dimensional simple representation of  $D(\mathcal{K}^{\text{cop}})$ . As  $a^4 = 1 = g^4$  and  $ga = ag$ ,  $\rho(a)$  and  $\rho(g)$  are simultaneously diagonalizable and we can assume that

$$\rho(a) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \rho(d) = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \quad \text{and} \quad \rho(g) = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix},$$

where  $\lambda_i^4 = 1$  for  $1 \leq i \leq 4$ . From  $ac = \xi ca$  we have that:

$$\begin{pmatrix} \lambda_1 c_{11} & \lambda_1 c_{12} \\ \lambda_2 c_{21} & \lambda_2 c_{22} \end{pmatrix} = \xi \begin{pmatrix} \lambda_1 c_{11} & \lambda_2 c_{12} \\ \lambda_1 c_{21} & \lambda_2 c_{22} \end{pmatrix}$$

what implies  $c_{11} = c_{22} = 0$ . Similarly, the relation  $gx = -xg$  implies  $x_{11} = x_{22} = 0$ . Since  $a^2c = b$ , we must have that

$$\rho(b) = \begin{pmatrix} 0 & \lambda_1^2 c_{12} \\ \lambda_2^2 c_{21} & 0 \end{pmatrix}.$$

Also note that from  $c^2 = 0$  we get  $c_{12}c_{21} = 0$ . Thus, by permuting the elements of the basis, we may assume that  $c_{21} = 0$ . Suppose  $c_{12} = 0$ . That is,

$$\rho(b) = \rho(c) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \rho(x) = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}.$$

Clearly, these modules are simple if and only if  $x_{12} \neq 0$  and  $x_{21} \neq 0$ . Since  $ax = -\xi xa + \sqrt{2}\xi(b + gc)$ , it follows that  $x_{12}(\lambda_1 + \xi\lambda_2) = 0$  and  $x_{21}(\lambda_2 + \xi\lambda_1) = 0$ . Since  $x_{12}x_{21} \neq 0$ , then  $\lambda_1 + \xi\lambda_2 = \lambda_2 + \xi\lambda_1 = 0$ , which implies that  $\lambda_1 = \lambda_2 = 0$  a contradiction.

Therefore, we must have that  $c_{12} \neq 0$ . Clearly, we may also assume that  $c_{12} = 1$ . From the equality  $ac = \xi ca$  we get that  $\lambda_2 = -\xi\lambda_1$  with  $\lambda_1 \neq 0$ . Moreover, since  $cg = -gc$ , it follows that  $\lambda_4 = -\lambda_3$ . Now, the relation  $ax + \xi xa = \sqrt{2}\xi(b + gc)$  yields

$$\begin{pmatrix} 0 & \lambda_1 x_{12} \\ \lambda_2 x_{21} & 0 \end{pmatrix} = -\xi \begin{pmatrix} 0 & \lambda_2 x_{12} \\ \lambda_1 x_{21} & 0 \end{pmatrix} + \sqrt{2}\xi \begin{pmatrix} 0 & \lambda_1^2 + \lambda_3 \\ 0 & 0 \end{pmatrix},$$

which implies that  $x_{12} = \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_3\lambda_1^{-1})$ , since  $\lambda_2 = -\xi\lambda_1$ . This is the same information obtained from  $dx - \xi xd = \sqrt{2}\xi(c + gb)$ . Analogously,  $cx + \xi xc = \sqrt{2}\xi(d - ga)$  yields

$$\begin{pmatrix} x_{21} & 0 \\ 0 & \xi x_{21} \end{pmatrix} = \sqrt{2}\xi \begin{pmatrix} \lambda_1^{-1} - \lambda_1\lambda_3 & 0 \\ 0 & \xi(\lambda_1^{-1} - \lambda_1\lambda_3) \end{pmatrix},$$

which gives  $x_{21} = \sqrt{2}\xi(\lambda_1^{-1} - \lambda_1\lambda_3)$ . This is the same information obtained from  $bx - \xi xb = \sqrt{2}\xi(a - gd)$ . Note that, since  $g^2 = 1 + x^2$ , we must have that  $x_{12}x_{21} = \lambda_3^2 - 1$ . In fact,

$$x_{12}x_{21} = \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_3\lambda_1^{-1})\sqrt{2}\xi(\lambda_1^{-1} - \lambda_1\lambda_3) = -(1 - \lambda_1^2\lambda_3 + \lambda_3\lambda_1^{-2} - \lambda_3^2) = \lambda_3^2 - 1.$$

From the discussion above, the matrices defining the action on the simple module  $V$  are of the form

$$\rho(a) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\xi\lambda_1 \end{pmatrix}, \rho(b) = \begin{pmatrix} 0 & \lambda_1^2 \\ 0 & 0 \end{pmatrix}, \rho(c) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \rho(g) = \begin{pmatrix} \lambda_3 & 0 \\ 0 & -\lambda_3 \end{pmatrix},$$

$$\rho(d) = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \xi\lambda_1^{-1} \end{pmatrix}, \rho(x) = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_3\lambda_1^{-1}) \\ \sqrt{2}\xi(\lambda_1^{-1} - \lambda_1\lambda_3) & 0 \end{pmatrix},$$

with  $\lambda_1^4 = 1 = \lambda_3^4$ . Moreover, a direct computation shows that  $V$  is simple if and only if  $\lambda_3 \neq \lambda_1^2$ . If we set  $\lambda_1 = \xi^i$  and  $\lambda_3 = \xi^j$  for some  $i, j \in \mathbb{Z}_4$ , we have that  $2i \neq j$  and consequently  $(i, j) \in \Lambda$ .

**Claim.**  $V_{i,j}$  is isomorphic to  $V_{k,\ell}$  if and only if  $(i, j) = (k, \ell)$  in  $\mathbb{Z}_4 \times \mathbb{Z}_4$ .

Let  $T : V_{i,j} \rightarrow V_{k,\ell}$  be an isomorphism of  $D$ -modules; in particular  $\rho_{k,\ell}(t)T = T\rho_{i,j}(t)$  for any  $t \in D$ . Denote by  $[T] = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$  the matrix of  $T$  in the given bases. Using the action of  $c$  we have that  $t_{21} = 0$  and  $t_{11} = t_{22}$  since

$$\begin{pmatrix} t_{21} & t_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & t_{11} \\ 0 & t_{21} \end{pmatrix}.$$

Moreover, acting by  $a$  we have

$$\begin{pmatrix} \xi^k t_{11} & \xi^k t_{12} \\ 0 & -\xi^{k+1} t_{11} \end{pmatrix} = \begin{pmatrix} \xi^k & 0 \\ 0 & -\xi^{k+1} \end{pmatrix} [T] = [T] \begin{pmatrix} \xi^i & 0 \\ 0 & -\xi^{i+1} \end{pmatrix} = \begin{pmatrix} \xi^i t_{11} & -\xi^{i+1} t_{12} \\ 0 & -\xi^{i+1} t_{11} \end{pmatrix}$$

which implies  $(\xi^k - \xi^i)t_{11} = 0$  and  $(\xi^k + \xi^{i+1})t_{12} = 0$ . Since  $T$  is an isomorphism, this implies that  $\xi^k = \xi^i$  from which follows that  $t_{12} = 0$  and consequently  $[T] = t_{11}I$ . Finally, acting by  $g$  yields that  $\xi^\ell = \xi^j$  and the claim follows.  $\square$

**Remark 2.8.** Let  $V$  be a left  $D$ -module. Since  $D$  is a Hopf algebra,  $V^*$  inherits a left  $D$ -module structure by the formula  $(h \cdot f)(v) = f(S(h) \cdot v)$  for all  $f \in V^*$ ,  $v \in V$  and  $h \in D$ . A straightforward calculation yields  $V_{i,j}^* \simeq V_{-i+1, j+2}$  for all  $(i, j) \in \mathbb{Z}_4 \times \mathbb{Z}_4$ .

Finally, we describe all simple left  $D$ -modules up to isomorphism.

**Theorem 2.9.** *There are 16 simple left  $D$ -modules pairwise non-isomorphic. Four one-dimensional given by Lemma 2.6 and 12 two-dimensional given by Lemma 2.7.*

*Proof.* Assume there is a simple module of dimension  $d > 2$  and let  $n$  denote the amount of simple  $d$ -dimensional modules pairwise non-isomorphic. By Lemmata 2.6 and 2.7 we have that

$$4 \cdot 1^2 + 12 \cdot 2^2 + nd^2 = 52 + nd^2 \leq \dim(D^*)_0 < \dim D^* = 64.$$

Then  $nd^2 < 12$ , which implies that  $d = 3$  and  $n = 1$ . But in such a case, by [AN, Lemma 2.1] we must have that  $4 = |G(D^*)|$  divides  $3^2 = 9$ , a contradiction.  $\square$

**2.3. Projective covers of simple left  $D$ -modules.** In this subsection we denote by  $\widehat{D}$  the set of isomorphism classes of simple left  $D$ -modules and by  $P(V)$  the projective cover of a simple  $D$ -module  $V$ . Projective covers are unique up to isomorphism and as left  $D$ -modules one has that

$${}_D D \simeq \bigoplus_{V \in \widehat{D}} P(V)^{\dim V}.$$

For  $j \in \mathbb{Z}_4$ , let  $\mathbb{k}_{\chi^j}$  be the one-dimensional  $D$ -module associated to the character  $\chi^j$ . Then  $1 < \dim P(\mathbb{k}_\varepsilon) \leq \dim V \dim P(V)$ , for all  $V \in \widehat{D}$ , see [EG, Section 2]. In particular, by item (ii) of the following lemma, it follows that  $\dim P(\mathbb{k}_\varepsilon) \leq 4$ .

**Lemma 2.10.** (i)  $V_{i,j} \otimes \mathbb{k}_{\chi^\ell} \simeq V_{i+\ell,j+2\ell}$  and  $\mathbb{k}_{\chi^\ell} \otimes \mathbb{k}_{\chi^k} \simeq \mathbb{k}_{\chi^{k+\ell}}$  for all  $(i,j) \in \Lambda$ ,  $k, \ell \in \mathbb{Z}_4$ .

(ii)  $P(V_{i,j}) = V_{i,j}$  for all  $(i,j) \in \Lambda$ .

(iii)  $P(\mathbb{k}_{\chi^\ell}) = P(\mathbb{k}_\varepsilon) \otimes \mathbb{k}_{\chi^\ell}$  and  $\dim P(\mathbb{k}_{\chi^\ell}) = 4$  for all  $\ell \in \mathbb{Z}_4$ .

*Proof.* (i) follows by a direct computation.

(ii) Let  $(i,j) \in \Lambda$  and  $\mu = \chi^\ell$  be a character of  $D$ . Since  $\text{Hom}(P(V_{i,j}) \otimes \mathbb{k}_\mu, V_{i,j} \otimes \mathbb{k}_\mu) = \text{Hom}(P(V_{i,j}), V_{i,j} \otimes \mathbb{k}_\mu \otimes \mathbb{k}_\mu^*) = \text{Hom}(P(V_{i,j}), V_{i,j}) \neq 0$ , and  $P(V_{i,j}) \otimes \mathbb{k}_\mu$  is projective, it follows that  $P(V_{i,j}) \otimes \mathbb{k}_\mu$  contains  $P(V_{i,j} \otimes \mathbb{k}_\mu) \simeq P(V_{i+\ell,j+2\ell})$ . If  $\dim P(V_{i,j}) > \dim V_{i,j}$  for some  $(i,j) \in \Lambda$ , then by [EO, Lemma 2.10] the socle of  $P(V_{i,j})$  is  $V_{i,j}$ , since  $D$  is unimodular. Thus,  $\dim P(V_{i,j}) \geq 2 \dim V_{i,j}$  and  $\dim P(V_{i-\ell,j-2\ell}) \geq \dim P(V_{i,j}) \geq 4$ . Thus, if we denote  $I = \{(m,n) \in \Lambda : (m,n) \neq (i+k,j+2k) \text{ for all } 0 \leq k \leq 3\}$  then

$$\dim D \geq \sum_{j=0}^3 \dim P(\mathbb{k}_{\chi^j}) + \sum_{(m,n) \in I} 2 \dim P(V_{m,n}) + 8 \dim P(V_{i,j}) \geq \sum_{j=0}^3 \dim P(\mathbb{k}_{\chi^j}) + 64,$$

a contradiction.

(iii) Since  $P(\mathbb{k}_\varepsilon)$  is projective,  $P(\mathbb{k}_\varepsilon) \otimes \mathbb{k}_{\chi^\ell}$  is also projective. As  $\text{Hom}(P(\mathbb{k}_\varepsilon) \otimes \mathbb{k}_{\chi^\ell}, \mathbb{k}_{\chi^\ell}) = \text{Hom}(P(\mathbb{k}_\varepsilon), \mathbb{k}_{\chi^\ell} \otimes \mathbb{k}_{\chi^\ell}^*) = \text{Hom}(P(\mathbb{k}_\varepsilon), \mathbb{k}_\varepsilon) \neq 0$ , it follows that  $P(\mathbb{k}_\varepsilon) \otimes \mathbb{k}_{\chi^\ell}$  contains  $P(\mathbb{k}_{\chi^\ell})$ , whose dimension is at least  $\dim P(\mathbb{k}_\varepsilon)$ . Hence they must be equal. Since  $P(V_{i,j}) = V_{i,j}$  for all  $(i,j) \in \Lambda$ , we have that  $64 = 4 \dim P(\mathbb{k}_\varepsilon) + 48$ , and whence  $\dim P(\mathbb{k}_\varepsilon) = 4$ .  $\square$

**Remark 2.11.** From Lemma 2.10 (i) and Remark 2.8 follows that  $V_{i,j}^* = V_{i,j} \otimes \mathbb{k}_\chi$  if  $i = 0, 2$  and  $(i,j) \in \Lambda$ .

Let  $P = \mathbb{k}\{p_1, p_2, p_3, p_4\}$  be the 4-dimensional  $D$ -module given by

$$(2) \quad \begin{array}{c|cccc} \cdot & p_1 & p_2 & p_3 & p_4 \\ \hline a & p_1 & -\xi p_2 & \xi p_3 & p_4 \\ \hline b & p_3 & \xi p_4 & 0 & 0 \\ \hline c & -p_3 & \xi p_4 & 0 & 0 \\ \hline d & p_1 & \xi p_2 & -\xi p_3 & p_4 \\ \hline x & p_2 + \sqrt{2}p_3 & -\sqrt{2}p_4 & p_4 & 0 \\ \hline g & p_1 & -p_2 & -p_3 & p_4 \end{array}$$

**Lemma 2.12.**  $P \simeq P(\mathbb{k}_\varepsilon)$  as  $D$ -modules.

*Proof.* A straightforward calculation shows that (2) endows  $P$  with a  $D$ -action. Moreover, consider the submodules given by  $\mathbb{k}p_4$  and  $W = \mathbb{k}\{p_2, p_3, p_4\}$ . Then  $\mathbb{k}p_4$  and  $P/W$  are both isomorphic to the trivial  $D$ -module  $\mathbb{k}_\varepsilon$ .

Assume that  $P = M \oplus N$  and let  $w = \alpha p_1 + \beta p_2 + \gamma p_3 + \delta p_4 \in M$ . If  $\alpha \neq 0$ , then  $xb \cdot w = \alpha p_4$  and  $p_4 \in M$ . Analogously,  $b \cdot w = \alpha p_3 + \xi \beta p_4$  which implies that  $p_3 \in M$ . Thus,  $w' = \alpha p_1 + \beta p_2 \in M$  and consequently  $x \cdot w' = \alpha(p_2 + \sqrt{2}p_3) - \sqrt{2}\beta p_4 \in M$  which gives us  $p_2 \in M$  and  $p_1 \in M$ , implying that  $N = 0$ . If  $\alpha = 0$ , then  $p_1 \in N$  and since  $Dp_1 = P$ , we have that  $N = P$ . Hence,  $P$  is indecomposable.

Finally, we show that  $P \simeq P(\mathbb{k}_\varepsilon)$ . Denote by  $\varphi : P \rightarrow \mathbb{k}_\varepsilon$  the epimorphism induced by the quotient  $P/W$ . Since  $P(\mathbb{k}_\varepsilon)$  is projective, there exists an epimorphism  $\pi$  such that the following diagram commutes

$$\begin{array}{ccc} & P(\mathbb{k}_\varepsilon) & \\ \pi \swarrow & \downarrow & \\ P & \xrightarrow{\varphi} & \mathbb{k}_\varepsilon. \end{array}$$

Thus, there exists an element  $w = \alpha p_1 + \beta p_2 + \gamma p_3 + \delta p_4 \in \pi(P(\mathbb{k}_\varepsilon))$  with  $\alpha \neq 0$ . Following the argument in the paragraph above we have that  $\pi$  is surjective and whence an isomorphism.  $\square$

As a consequence of the results above we have the following theorem.

**Theorem 2.13.** *The  $D$ -modules  $P_\ell = P \otimes \mathbb{k}_{\chi_\ell}$  and  $V_{i,j}$  with  $i, j, \ell \in \mathbb{Z}_4$  and  $2i \neq j$  are the projective covers of the simple  $D$ -modules and consequently*

$${}_D D \simeq \sum_{\ell=0}^3 P_\ell \oplus \sum_{(i,j) \in \Lambda} V_{i,j}^2.$$

$\square$

For  $0 \leq \ell \leq 3$ , denote by  $\{p_{i,\ell} = p_i \otimes 1\}_{1 \leq i \leq 4}$  the linear basis of  $P_\ell$ , with  $p_{i,0} = p_i$ . Using (2) and the following equalities, the  $D$ -module structure of  $P_\ell$  can be described explicitly.

$$\begin{aligned} (3) \quad & a \cdot (p_{i,\ell}) = a \cdot (p_i \otimes 1) = (a \cdot p_i) \otimes (a \cdot 1) + (b \cdot p_i) \otimes (c \cdot 1) = \xi^\ell(a \cdot p_i) \otimes 1 \\ & b \cdot (p_{i,\ell}) = b \cdot (p_i \otimes 1) = (a \cdot p_i) \otimes (b \cdot 1) + (b \cdot p_i) \otimes (d \cdot 1) = \xi^{-\ell}(b \cdot p_i) \otimes 1 \\ & c \cdot (p_{i,\ell}) = c \cdot (p_i \otimes 1) = (c \cdot p_i) \otimes (a \cdot 1) + (d \cdot p_i) \otimes (c \cdot 1) = \xi^\ell(c \cdot p_i) \otimes 1 \\ & d \cdot (p_{i,\ell}) = g \cdot (p_i \otimes 1) = (c \cdot p_i) \otimes (b \cdot 1) + (d \cdot p_i) \otimes (d \cdot 1) = \xi^{-\ell}(d \cdot p_i) \otimes 1 \\ & x \cdot (p_{i,\ell}) = x \cdot (p_i \otimes 1) = (x \cdot p_i) \otimes (g \cdot 1) + (1 \cdot p_i) \otimes (x \cdot 1) = (-1)^\ell(x \cdot p_i) \otimes 1 \\ & g \cdot (p_{i,\ell}) = g \cdot (p_i \otimes 1) = (g \cdot p_i) \otimes (g \cdot 1) = (-1)^\ell(g \cdot p_i) \otimes 1. \end{aligned}$$

**2.4. Some indecomposable  $D$ -modules.** Let  $A$  be a finite-dimensional  $\mathbb{k}$ -algebra and  $V_1, \dots, V_n$  a complete list of non-isomorphic simple left  $A$ -modules. The Ext-Quiver of  $A$  is the quiver  $\text{Ext}Q(A)$  with vertices  $1, \dots, n$  and  $\dim \text{Ext}_A^1(V_i, V_j)$  arrows from the vertex  $i$  to the vertex  $j$ . Given a quiver  $Q$  with vertices  $1, \dots, n$ , its separation diagram is the unoriented graph with vertices  $1, \dots, n, 1', \dots, n'$  and with an edge from  $i$  to  $j'$  for each arrow  $i \rightarrow j$  in  $Q$ . The separation diagram of  $A$  is the separation diagram of its Ext-Quiver. It is well-known that a finite-dimensional algebra is of finite (tame) representation type if and only if its separation diagram is a disjoint union of finite (affine) Dynkin diagrams.

In this section we compute the separation diagram of  $D$  and show that  $D$  is of tame representation type. In order to do so, we use the isomorphism of abelian groups between  $\dim \text{Ext}_A^1(V_i, V_j)$  and equivalence classes of extensions  $0 \rightarrow V_j \rightarrow M \rightarrow V_i \rightarrow 0$  of  $V_i$  by  $V_j$ ; here, the neutral element is given by the trivial extension.

**2.4.1. 2-dimensional (non-simple) indecomposable modules.** Let  $A$  be the subalgebra of  $D$  generated by  $a, d$  and  $g$ . Then  $A$  is an 8-dimensional commutative algebra given by  $A = \mathbb{k}\langle a, g : a^4 = 1 = g^4 \rangle$ . In particular, all simple modules are one-dimensional and the restriction to  $A$  of the characters of  $D$  induce characters on  $A$ .

**Definition 2.14.** Let  $0 \leq \ell \leq 3$ . Define  $M_\ell^+ = \mathbb{k}\{m_1, m_2\}$  to be the 2-dimensional  $D$ -module given by  $\mathbb{k}m_1 \simeq \mathbb{k}_{\chi^\ell}$  and

$$\begin{aligned} a \cdot m_2 &= \chi^{\ell+1}(a) m_2 = \xi^{\ell+1} m_2, & b \cdot m_2 &= 0 = c \cdot m_2, \\ g \cdot m_2 &= \chi^{\ell+1}(g) m_2 = (-1)^{\ell+1} m_2, & x \cdot m_2 &= m_1. \end{aligned}$$

Then,  $M_\ell^+$  is an indecomposable module containing  $\mathbb{k}_{\chi^\ell}$  and  $M_\ell^+/\mathbb{k}_{\chi^\ell} = \mathbb{k}_{\chi^{\ell+1}}$ . Analogously, define  $M_\ell^- = \mathbb{k}\{m_1, m_2\}$  to be the 2-dimensional  $D$ -module given by  $\mathbb{k}m_1 \simeq \mathbb{k}_{\chi^\ell}$  and

$$\begin{aligned} a \cdot m_2 &= \xi^{\ell-1} m_2, & b \cdot m_2 &= \frac{\sqrt{2}}{2} \xi^{\ell-1} m_1, & c \cdot m_2 &= \frac{\sqrt{2}}{2} (-\xi)^{\ell-1} m_1, \\ g \cdot m_2 &= (-1)^{\ell-1} m_2, & x \cdot m_2 &= m_1. \end{aligned}$$

Then,  $M_\ell^-$  is an indecomposable module containing  $\mathbb{k}_{\chi^\ell}$  and  $M_\ell^-/\mathbb{k}_{\chi^\ell} = \mathbb{k}_{\chi^{\ell-1}}$ .

**Lemma 2.15.** Let  $0 \leq \ell \leq 3$ .

- (i) Let  $M$  be a 2-dimensional indecomposable module containing  $\mathbb{k}_{\chi^\ell}$ . Then  $M \simeq M_\ell^+$  or  $M \simeq M_\ell^-$ .
- (ii)

$$\dim \text{Ext}_D^1(\mathbb{k}_{\chi^k}, \mathbb{k}_{\chi^\ell}) = \begin{cases} 1 & \text{if } k = \ell \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* (i) Assume  $M$  is a 2-dimensional indecomposable  $D$ -module containing  $\mathbb{k}_\lambda$  with  $\lambda = \chi^\ell$ . We must have that  $M \simeq \mathbb{k}_\lambda \oplus \mathbb{k}_\mu$  as  $A$ -modules, with  $\mu$  some character on  $D$ . That is,  $M$  has a linear basis  $\{m_1, m_2\}$  such that  $\mathbb{k}m_1 \simeq \mathbb{k}_\lambda$ , and  $a \cdot m_2 = \mu(a)m_2$ ,  $d \cdot m_2 = \mu(d)m_2$  and  $g \cdot m_2 = \mu(g)m_2$ , with  $\mu(a)^2 = \mu(g)$ . In particular,  $M$  fits into an exact sequence of the form

$$0 \rightarrow \mathbb{k}_\lambda \rightarrow M \rightarrow \mathbb{k}_\mu \rightarrow 0.$$

Moreover, since  $M/\mathbb{k}_\lambda \simeq \mathbb{k}_\mu$ , we must have that

$$b \cdot m_2 = \beta m_1, \quad c \cdot m_2 = \gamma m_1 \quad \text{and} \quad x \cdot m_2 = \alpha m_1,$$

for some  $\beta, \gamma, \alpha \in \mathbb{k}$ . Since  $a^2b = c$ , then  $\lambda(a)^2\beta = \gamma$ . Using the relations  $cg = -gc$ ,  $bg = -gb$  and  $xg = -gx$  we obtain the equalities

$$\beta(\lambda(g) + \mu(g)) = 0, \quad \gamma(\lambda(g) + \mu(g)) = 0 \quad \text{and} \quad \alpha(\lambda(g) + \mu(g)) = 0.$$

Hence, if  $\mu(g) \neq -\lambda(g)$  then  $\beta = 0$ ,  $\gamma = 0$  and  $\alpha = 0$  which implies that  $M \simeq \mathbb{k}_\lambda \oplus \mathbb{k}_\mu$ .

Assume that  $\mu(g) = -\lambda(g)$ . From the relation  $bx - \xi xb = \sqrt{2}\xi(a - gd)$  we deduce that  $\lambda(g) = \lambda(a)^2$  and  $\mu(g) = \mu(a)^2$ . Thus, we must have that  $\mu(a)^2 = -\lambda(a)^2$  and consequently  $\mu(a) = \pm \xi \lambda(a)$ . Note that the relation  $cx + \xi xc = \sqrt{2}\xi(d - ga)$  yields the same conclusion. On the other hand, the relations  $ax + \xi xa = \sqrt{2}\xi(b + gc)$  and  $dx - \xi xd = \sqrt{2}\xi(c + gb)$  give the equations

$$\alpha(\lambda(a) + \xi\mu(a)) = \sqrt{2}\xi 2\beta, \quad \alpha(\lambda(d) - \xi\mu(d)) = \sqrt{2}\xi 2\lambda(a)^2\beta.$$

Using that  $\mu(a)^2 = -\lambda(a)^2$  and  $\lambda(d) = \lambda(a)^3$ , it follows that the second equality is the first equality multiplied by  $\lambda(a)^2$ .

If  $\mu(a) = \xi\lambda(a)$ , then  $\mu = \chi^{\ell+1}$  and  $\lambda(a) + \xi\mu(a) = 0$ , which implies that  $\beta = 0 = \gamma$ . In this case,  $M$  is an indecomposable module isomorphic to  $M_\ell^+$ . Denote this module by  $M_\ell^+(\alpha)$ .

If  $\mu(a) = -\xi\lambda(a)$ , then  $\mu = \chi^{\ell+3} = \chi^{\ell-1}$  and  $\lambda(a) + \xi\mu(a) = 2\lambda(a)$ , which implies that  $\alpha\lambda(a) = \sqrt{2}\xi\beta$ . In this case,  $M$  is isomorphic to  $M_\ell^-$  and (i) follows. Denote this module by  $M_\ell^-(\alpha)$ .

(ii) By the preceding discussion, we have that  $\dim \text{Ext}_D^1(\mathbb{k}_{\chi^k}, \mathbb{k}_{\chi^\ell}) = 0$  if  $k \neq \ell \pm 1$ . Let  $M$  be an extension of  $\mathbb{k}_{\chi^\ell}$  by  $\mathbb{k}_{\chi^{\ell \pm 1}}$ . Then,  $M = M_\ell^\pm(\alpha)$  for some  $\alpha \in \mathbb{k}^\times$ . Let  $\alpha, \alpha' \in \mathbb{k}^\times$  and assume  $M_\ell^\pm(\alpha) \simeq M_\ell^\pm(\alpha')$  as extensions. If  $\{m_1, m_2\}$  and  $\{m'_1, m'_2\}$  denote



the linear basis of  $M_\ell^\pm(\alpha)$  and  $M_\ell^\pm(\alpha')$ , respectively, and  $\varphi$  the isomorphism, we must have that  $\varphi(m_1) = m'_1$  and  $\varphi(m_2) = \gamma m'_1 + \eta m'_2$  with  $\eta \neq 0$ . Moreover,  $\varphi(x \cdot m_2) = \alpha \varphi(m_2) = \alpha m'_1$  must be equal to  $x \cdot \varphi(m_2) = \eta \alpha' m'_1$ . Hence,  $\alpha = \eta \alpha'$  implying that  $\dim \text{Ext}_D^1(\mathbb{k}_{\chi^{\ell+1}}, \mathbb{k}_{\chi^\ell}) = 1$ .  $\square$

**2.4.2. 3-dimensional indecomposable modules.** In this subsection we describe the 3-dimensional indecomposable modules as we did in the previous subsection for dimension 2. First, we prove that  $\dim \text{Ext}_D^1(V, W) = 0$  for all simple modules  $V, W$  such that  $\dim V \dim W = 2$ . In particular, in the Ext-Quiver of  $D$ , the component of vertices corresponding to one-dimensional modules is disconnected to the one corresponding to 2-dimensional modules.

**Lemma 2.16.** (i)  $\dim \text{Ext}_D^1(V_{i,j}, V_{k,\ell}) = 0$  for all  $(i, j), (k, \ell) \in \Lambda$ .

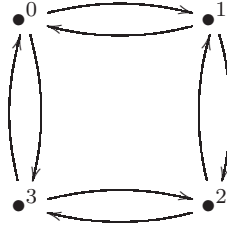
(ii)  $\dim \text{Ext}_D^1(V_{i,j}, \mathbb{k}_{\chi^\ell}) = 0 = \dim \text{Ext}_D^1(\mathbb{k}_{\chi^\ell}, V_{i,j})$  for all  $(i, j) \in \Lambda$  and  $\ell \in \mathbb{Z}_4$ .

*Proof.* (i) follows immediately since  $V_{i,j}$  is projective for all  $(i, j) \in \Lambda$ .

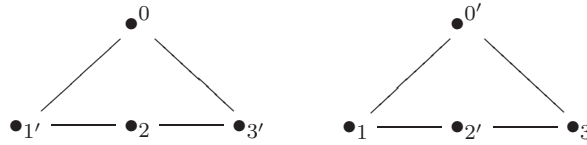
(ii) By taking duals, we have that  $\dim \text{Ext}_D^1(V, W) = \dim \text{Ext}_D^1(W^*, V^*)$ . Thus, it is enough to prove the assertion for  $V = V_{i,j}$  and  $W = \mathbb{k}_{\chi^\ell}$  for  $(i, j) \in \Lambda$  and  $\ell \in \mathbb{Z}_4$ . Since  $V_{i,j}$  is projective for every  $(i, j) \in \Lambda$  by Lemma 2.10 (ii), the claim follows.  $\square$

**Corollary 2.17.**  $D$  is of tame representation type.

*Proof.* Denote by  $i$  the vertex corresponding to the character  $\chi^i$  for all  $0 \leq i \leq 3$ . Lemma 2.15 implies that  $\text{ExtQ}(D)$  contains the quiver



Thus, the separation diagram of  $D$  contains the quiver  $A_3^{(1)} \amalg A_3^{(1)}$



Moreover, by Lemma 2.16,  $\text{ExtQ}(D)$  consists of the quiver above and isolated points representing the simple modules  $V_{i,j}$ . Hence,  $D$  is of tame representation type.  $\square$

**Remark 2.18.** Notice that, since the simple representations  $V_{i,j}$  are projective for all  $(i, j) \in \Lambda$ , they cannot be contained in the socle or the top of any non-simple indecomposable module  $M$ . Hence,  $\text{Soc}(M)$  and  $\text{Top}(M)$  consist of direct sums of one-dimensional modules.

Furthermore, if  $0 \subseteq \text{Soc}(M) \subseteq \text{Soc}^2(M) \subseteq \dots \subseteq \text{Soc}^n(M) = M$  denotes the socle series of  $M$ , then  $\text{Soc}(M/\text{Soc}^i(M))$  does not contain any simple projective module. Indeed, assume that  $\text{Soc}(M/\text{Soc}^i(M)) = \text{Soc}^{i+1}(M)/\text{Soc}^i(M)$  contains a simple projective module  $S$ , and denote by  $j : S \rightarrow \text{Soc}^{i+1}(M)/\text{Soc}^i(M)$  the inclusion and by  $p : \text{Soc}^{i+1}(M) \rightarrow \text{Soc}^{i+1}(M)/\text{Soc}^i(M)$  the projection. Since  $S$  is projective, there exists a morphism  $t : S \rightarrow \text{Soc}^{i+1}(M)$  such that  $p \circ t = j$ . Since  $S$  is simple,  $t$  is a monomorphism and whence  $\text{Soc}^{i+1}(M)$  contains a simple module  $t(S)$  isomorphic to  $S$ . But  $t(S) \subseteq \text{Soc}(\text{Soc}^{i+1}(M)) \subseteq \text{Soc}(M) \subseteq \text{Soc}^i(M)$ , which implies that  $j(S) = p(t(S)) = 0$ , a contradiction.

In the following, we determine the 3-dimensional indecomposable modules.

**Definition 2.19.** Let  $N_\ell = \mathbb{k}\{n_1, n_2, n_3\}$  be the 3-dimensional  $D$ -module given by  $\mathbb{k}n_1 \simeq \mathbb{k}_{\chi^\ell}$ ,  $\mathbb{k}n_2 \simeq \mathbb{k}_{\chi^{\ell+2}}$  and

$$\begin{aligned} a \cdot n_3 &= \xi^{\ell+1} n_3, & b \cdot n_3 &= \frac{\sqrt{2}}{2} \xi^{\ell+1} n_2, & c \cdot n_3 &= -\frac{\sqrt{2}}{2} (-\xi)^{\ell+1} n_2, \\ g \cdot n_3 &= (-1)^{\ell+1} n_3, & x \cdot n_3 &= n_1 + n_2. \end{aligned}$$

Then,  $N_\ell$  is an indecomposable module with socle  $\mathbb{k}_{\chi^\ell} \oplus \mathbb{k}_{\chi^{\ell+2}}$ ,  $N_\ell/(\mathbb{k}_{\chi^\ell} \oplus \mathbb{k}_{\chi^{\ell+2}}) = \mathbb{k}_{\chi^{\ell+1}}$ ,  $N_\ell/\mathbb{k}_{\chi^\ell} = M_\ell^+$  and  $N_\ell/\mathbb{k}_{\chi^{\ell+2}} = M_\ell^-$ .

**Lemma 2.20.** Let  $N$  be a 3-dimensional indecomposable  $D$ -module. Then  $N \simeq N_\ell$  for some  $0 \leq \ell \leq 3$ .

*Proof.* By Remark 2.18,  $\text{Soc}(N)$  contains only one-dimensional modules. If  $\text{Soc}(N) = \mathbb{k}_\lambda$  for some  $D$ -character  $\lambda$ , then  $N$  is injective by [ARS, Proposition II. 4.1 (d)]. But in such a case,  $N^*$  would be a projective module with an epimorphism to  $\mathbb{k}_{\lambda^{-1}}$  and whence  $\dim N^* \geq \dim P(\mathbb{k}_{\lambda^{-1}}) = 4$ , a contradiction. Hence,  $\text{Soc}(N) = \mathbb{k}_\lambda \oplus \mathbb{k}_\mu$  for some  $D$ -characters  $\lambda, \mu$  and  $N$  fits into an exact sequence

$$(4) \quad 0 \rightarrow \mathbb{k}_\lambda \oplus \mathbb{k}_\mu \rightarrow N \rightarrow \mathbb{k}_\tau \rightarrow 0,$$

for some  $D$ -character  $\tau$ . In particular,  $N \simeq \mathbb{k}_\lambda \oplus \mathbb{k}_\mu \oplus \mathbb{k}_\tau$  as  $A$ -modules. Let  $\{n_1, n_2, n_3\}$  be a linear basis of  $N$  such that  $\mathbb{k}n_1 \simeq \mathbb{k}_\lambda$ ,  $\mathbb{k}n_2 \simeq \mathbb{k}_\mu$  and

$$\begin{aligned} a \cdot n_3 &= \tau(a)n_3, & d \cdot n_3 &= \tau(d)n_3, & g \cdot n_3 &= \tau(g)n_3 = \tau(a)^2 n_3, \\ b \cdot n_3 &= \beta_1 n_1 + \beta_2 n_2, & c \cdot n_3 &= \gamma_1 n_1 + \gamma_2 n_2, & x \cdot n_3 &= \theta_1 n_1 + \theta_2 n_2. \end{aligned}$$

As  $a^2b = c$ , we have that  $\lambda(a)^2\beta_1 = \gamma_1$  and  $\mu(a)^2\beta_2 = \gamma_2$ . Further, using the relations  $cg = -gc$ ,  $bg = -gb$ ,  $xg = -gx$  and  $ax + \xi xa = \sqrt{2}\xi(b + gc)$  we obtain the equalities

$$\begin{aligned} \beta_1(\lambda(g) + \tau(g)) &= 0, & \beta_2(\mu(g) + \tau(g)) &= 0, \\ \gamma_1(\lambda(g) + \tau(g)) &= 0, & \gamma_2(\mu(g) + \tau(g)) &= 0, \\ \theta_1(\lambda(g) + \tau(g)) &= 0, & \theta_2(\mu(g) + \tau(g)) &= 0, \\ \theta_1(\lambda(a) + \xi\tau(a)) &= 2\sqrt{2}\xi\beta_1, & \theta_2(\mu(a) + \xi\tau(a)) &= 2\sqrt{2}\xi\beta_2. \end{aligned}$$

If  $\tau(g) \neq -\lambda(g)$  and  $\tau(g) \neq -\mu(g)$ , then  $\beta_i = \gamma_i = \theta_i = 0$  and consequently  $N \simeq \mathbb{k}_\lambda \oplus \mathbb{k}_\mu \oplus \mathbb{k}_\tau$  as  $D$ -modules, a contradiction. If  $\tau(g) = -\lambda(g)$  but  $\tau(g) \neq -\mu(g)$ , then  $\beta_2 = \gamma_2 = \theta_2 = 0$  and this implies that  $N = L \oplus \mathbb{k}_\mu$  with  $L = \mathbb{k}\{n_1, n_3\}$ . Analogously,  $N$  is decomposable if  $\tau(g) = -\mu(g)$  but  $\tau(g) \neq -\lambda(g)$ . Hence,  $-\tau(g) = \lambda(g) = \mu(g)$  and thus  $\lambda(a) = \pm\mu(a) = \pm\xi\tau(a)$ . The same reasoning shows that  $\theta_1 \neq 0 \neq \theta_2$  since otherwise  $N$  would be decomposable. So, we may assume that  $\theta_1 = \theta_2 = 1$ .

Moreover,  $\lambda = -\mu = \xi^2\mu$  since otherwise  $N$  is also decomposable. Indeed, if  $\lambda = \mu$ , let  $v = n_1 + n_2$ . Then  $\mathbb{k}v \simeq \mathbb{k}_\lambda$  and  $N \simeq \mathbb{k}_\lambda \oplus \mathbb{k}\{v, n_3\}$  as  $D$ -modules.

Let  $\lambda = \chi^\ell$  for some  $0 \leq \ell \leq 3$ . Then  $\mu = \chi^{\ell+2}$  and  $\tau = \chi^{\ell\pm 3} = \chi^{\ell\pm 1}$ . From the last equation above it follows that if  $\tau(a) = \chi^{\ell+1}(a) = \xi^{\ell+1}$ , then  $\beta_1 = 0$  and if  $\tau(a) = \chi^{\ell-1}(a) = \xi^{\ell-1}$ , then  $\beta_2 = 0$ . Assume  $\tau = \chi^{\ell+1}$ , then  $\beta_2 = \frac{\sqrt{2}}{2}\xi^{\ell+1}$  and  $\gamma_2 = -\frac{\sqrt{2}}{2}(-\xi)^{\ell+1}$ . In such a case,  $N \simeq N_\ell$ . If  $\tau = \chi^{\ell-1}$ , the same argument also shows that  $N \simeq N_\ell$  and the lemma is proved.  $\square$

### 3. THE CATEGORY ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$

Using the equivalence  ${}_D\mathcal{M} \simeq {}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$  we determine in this section the simple and some indecomposable objects of the latter and we describe their braidings. Note that by [AG, Proposition 2.2.1], one has  ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD} \simeq {}_{\mathcal{A}}^{\mathcal{A}}\mathcal{YD}$  with  $\mathcal{A} = \mathcal{A}_4'' = \mathcal{K}^*$ .

**3.1. Simple objects and projective covers in  ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$ .** Our intention is to describe the simple  $D$ -modules and their projective covers as left Yetter-Drinfeld modules over  $\mathcal{K}$ . To do so, we simply need to describe the coaction of  $\mathcal{K}$ .

**Proposition 3.1.** *Let  $\mathbb{k}_{\chi^j} = \mathbb{k}v$  be a one-dimensional  $D$ -module with  $j \in \mathbb{Z}_4$ . Then  $\mathbb{k}_{\chi^j} \in {}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$  with its structure given by*

$$a \cdot v = \xi^j v, \quad b \cdot v = c \cdot v = 0, \quad d \cdot v = \xi^{-j} v, \quad \delta(v) = a^{2j} \otimes v.$$

*Proof.* The action is given by the restriction of the action given in Lemma 2.6. Since  $\mathbb{k}_{\chi^j}$  is one-dimensional, we must have that  $\delta(v) = h \otimes v$  with  $h \in G(\mathcal{K}) = \{1, a^2\}$ . As  $f \cdot v = \langle f, h \rangle v$  for all  $f \in \mathcal{K}^*$  and  $\langle g, a^2 \rangle = -1$ , it follows that  $\delta(v) = a^{2j} \otimes v$ .  $\square$

The following proposition gives the braiding of  $\mathbb{k}_{\chi^j}$  for all  $j \in \mathbb{Z}_4$ .

**Proposition 3.2.** *The braiding of the one-dimensional Yetter-Drinfeld module  $\mathbb{k}_{\chi^j}$  is  $c(v \otimes v) = (-1)^j v \otimes v$ .*

*Proof.* By formula (1) and Proposition 3.1 we have

$$c(v \otimes v) = a^{2j} \cdot v \otimes v = (\xi^j)^{2j} v \otimes v = (-1)^{j^2} v \otimes v = (-1)^j v \otimes v.$$

$\square$

**Proposition 3.3.** *Let  $V_{i,j} = \mathbb{k}\{e_1, e_2\}$  be a two-dimensional simple  $D$ -module with  $(i, j) \in \Lambda$ . If we denote  $\lambda_1 = \xi^i$  and  $\lambda_2 = \xi^j$ , then  $V_{i,j} \in {}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$  with its action given by*

$$\begin{aligned} a \cdot e_1 &= \lambda_1 e_1, & b \cdot e_1 &= 0, & c \cdot e_1 &= 0, & d \cdot e_1 &= \lambda_1^{-1} e_1, \\ a \cdot e_2 &= -\xi \lambda_1 e_2, & b \cdot e_2 &= \lambda_1^2 e_1, & c \cdot e_2 &= e_1, & d \cdot e_2 &= \xi \lambda_1^{-1} e_2, \end{aligned}$$

and its coaction by

$$\begin{aligned} \delta(e_1) &= 1 \otimes e_1 - 2\lambda_1 ac \otimes e_2, & \delta(e_2) &= a^2 \otimes e_2, & \text{for } \lambda_2 &= 1, \\ \delta(e_1) &= a^2 \otimes e_1 + 2\lambda_1 ab \otimes e_2, & \delta(e_2) &= 1 \otimes e_2, & \text{for } \lambda_2 &= -1, \\ \delta(e_1) &= d \otimes e_1 + (\lambda_1^3 - \xi \lambda_1) c \otimes e_2, & \delta(e_2) &= a \otimes e_2 + \frac{1}{2}(\lambda_1 + \xi \lambda_1^3) b \otimes e_1, & \text{for } \lambda_2 &= \xi, \\ \delta(e_1) &= a \otimes e_1 + (\lambda_1^3 + \xi \lambda_1) b \otimes e_2, & \delta(e_2) &= d \otimes e_2 + \frac{1}{2}(\lambda_1 - \xi \lambda_1^3) c \otimes e_1 & \text{for } \lambda_2 &= -\xi. \end{aligned}$$

*Proof.* We just need to describe the coaction. Using that  $\delta(v) = \sum_{i=1}^8 v_i \otimes v^i \cdot v$  with  $\{v_i\}_{1 \leq i \leq 8}$  a basis of  $\mathcal{K}$  and  $\{v^i\}_{1 \leq i \leq 8}$  its dual basis, and the isomorphism given in Lemma 2.3 have that

$$\begin{aligned} \delta(e_1) &= \sum_{i=0}^3 (g^i)^* \otimes g^i \cdot e_1 + \sum_{i=0}^3 (xg^i)^* \otimes xg^i \cdot e_1 \\ &= \sum_{i=0}^3 (g^i)^* \otimes \lambda_2^i e_1 + \sum_{i=0}^3 (xg^i)^* \otimes (-\lambda_2)^i x_{21} e_2, \\ \delta(e_2) &= \sum_{i=0}^3 (g^i)^* \otimes g^i \cdot e_2 + \sum_{i=0}^3 (xg^i)^* \otimes xg^i \cdot e_2 \\ &= \sum_{i=0}^3 (g^i)^* \otimes (-\lambda_2)^i e_2 + \sum_{i=0}^3 (xg^i)^* \otimes \lambda_2^i x_{12} e_1, \end{aligned}$$

where  $(g^i)^* = \frac{1}{4}(1 + \xi^i a + (-\xi)^i d + (-1)^i a^2)$ ,  $(xg^i)^* = \frac{1}{4\sqrt{2}\xi}((-\xi)^i b + \xi^i c + ab + (-1)^i ac)$

for all  $0 \leq i \leq 3$ ,  $x_{21} = \sqrt{2}\xi(\lambda_1^3 - \lambda_1 \lambda_2)$ ,  $x_{12} = \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_1^3 \lambda_2)$ . Explicitly, if  $j = 0$  and  $i \in$

$\{1, 3\}$  then  $\lambda_2 = 1$ ,  $\lambda_1 = \pm\xi$  and  $x_{21} = \sqrt{2}\xi(\lambda_1^3 - \lambda_1) = -2\sqrt{2}\xi\lambda_1$ ,  $x_{12} = \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_1^3) = 0$ .

In such a case,  $\delta(e_1) = 1 \otimes e_1 - 2\lambda_1 ac \otimes e_2$  and  $\delta(e_2) = a^2 \otimes e_2$ .

If  $j = 2$  and  $i \in \{0, 2\}$ , then  $\lambda_2 = -1$ ,  $\lambda_1 = \pm 1$  and  $x_{21} = \sqrt{2}\xi(\lambda_1^3 + \lambda_1) = 2\sqrt{2}\xi\lambda_1$ ,  $x_{12} = \frac{\sqrt{2}}{2}\xi(\lambda_1 - \lambda_1^3) = 0$ . In such a case,  $\delta(e_1) = a^2 \otimes e_1 + 2\lambda_1 ab \otimes e_2$  and  $\delta(e_2) = 1 \otimes e_2$ .

If  $j = 1$  and  $i$  is arbitrary, then  $\lambda_2 = \xi$ , and  $\delta(e_1) = d \otimes e_1 + (\lambda_1^3 - \xi\lambda_1)c \otimes e_2$  and  $\delta(e_2) = a \otimes e_2 + \frac{1}{2}(\lambda_1 + \xi\lambda_1^3)b \otimes e_1$ .

Finally, if  $j = 3$  and  $i$  is arbitrary, then  $\lambda_2 = -\xi$  and  $\delta(e_1) = a \otimes e_1 + (\lambda_1^3 + \xi\lambda_1)b \otimes e_2$  and  $\delta(e_2) = d \otimes e_2 + \frac{1}{2}(\lambda_1 - \xi\lambda_1^3)c \otimes e_1$ .  $\square$

Next we describe the braiding of the simple modules  $V_{i,j}$  in  ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$ . To do so, we use a matrix-like notation to describe it in a compact way. Its proof follows by a direct computation using formula (1) and Proposition 3.3.

**Proposition 3.4.** *Denote as above  $\lambda_1 = \xi^i$  and  $\lambda_2 = \xi^j$ . The braiding of  $V_{i,j} \in {}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$  is given by the following formulae:*

(i) *If  $j = 0$ ,  $i \in \{1, 3\}$ , then*

$$c\left(\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix}\right) \otimes \begin{Bmatrix} e_1 & e_2 \end{Bmatrix} = \begin{Bmatrix} e_1 \otimes e_1 & e_2 \otimes e_1 + 2e_1 \otimes e_2 \\ -e_1 \otimes e_2 & e_2 \otimes e_2 \end{Bmatrix}.$$

(ii) *If  $j = 2$ ,  $i \in \{0, 2\}$ , then*

$$c\left(\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix}\right) \otimes \begin{Bmatrix} e_1 & e_2 \end{Bmatrix} = \begin{Bmatrix} e_1 \otimes e_1 & -e_2 \otimes e_1 + 2e_1 \otimes e_2 \\ e_1 \otimes e_2 & e_2 \otimes e_2 \end{Bmatrix}.$$

(iii) *If  $j = 1$  and  $i$  is arbitrary, then*

$$c\left(\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix}\right) \otimes \begin{Bmatrix} e_1 & e_2 \end{Bmatrix} = \begin{Bmatrix} \lambda_1^3 e_1 \otimes e_1 & \xi \lambda_1^3 e_2 \otimes e_1 + (\lambda_1^3 - \xi \lambda_1) e_1 \otimes e_2 \\ \lambda_1 e_1 \otimes e_2 & -\xi \lambda_1 e_2 \otimes e_2 + \frac{1}{2}(\lambda_1^3 + \xi \lambda_1) e_1 \otimes e_1 \end{Bmatrix}.$$

(iv) *If  $j = 3$  and  $i$  is arbitrary, then*

$$c\left(\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix}\right) \otimes \begin{Bmatrix} e_1 & e_2 \end{Bmatrix} = \begin{Bmatrix} \lambda_1 e_1 \otimes e_1 & -\xi \lambda_1 e_2 \otimes e_1 + (\lambda_1 + \xi \lambda_1^3) e_1 \otimes e_2 \\ \lambda_1^3 e_1 \otimes e_2 & \xi \lambda_1^3 e_2 \otimes e_2 + \frac{1}{2}(\lambda_1 - \xi \lambda_1^3) e_1 \otimes e_1 \end{Bmatrix}.$$

$\square$

**Remark 3.5.** *All the braidings given by Proposition 3.4 are not of diagonal type. See the Appendix for a proof.*

We end this section with the description of the projective covers of the one-dimensional modules  $\mathbb{k}_{\chi^j}$  for  $j \in \mathbb{Z}_4$  as objects in  ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$ .

**Proposition 3.6.** *Let  $P_j$  be the projective cover of the one-dimensional  $D$ -module  $\mathbb{k}_{\chi^j}$ , with  $j \in \mathbb{Z}_4$ . Then  $P_j \in {}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$  with its action given by (2) and (3) and its coaction by*

$$\delta(p_{1,j}) = (a^2)^j \otimes p_{1,j} - \frac{\xi\sqrt{2}}{2}(-a^2)^j ac \otimes (p_{2,j} + \sqrt{2}p_{3,j}),$$

$$\delta(p_{2,j}) = (a^2)^{j+1} \otimes p_{2,j} + \xi(-a^2)^j ab \otimes p_{4,j},$$

$$\delta(p_{3,j}) = (a^2)^{j+1} \otimes p_{3,j} - \frac{\xi\sqrt{2}}{2}(-a^2)^j ab \otimes p_{4,j},$$

$$\delta(p_{4,j}) = (a^2)^j \otimes p_{4,j}.$$

*Proof.* The action of  $\mathcal{K}$  is given by the restriction of the action of  $D$ . To describe the coaction we use again that for all  $p \in P_j$  we have that  $\delta(p) = \sum_{i=1}^8 v_i \otimes v^i \cdot p$  with

$\{v_i\}_{1 \leq i \leq 8}$  a basis of  $\mathcal{K}$ , and  $\{v^i\}_{1 \leq i \leq 8}$  its dual basis. Using the isomorphism given in Lemma 2.3, it follows that

$$\begin{aligned}
\delta(p_{1,j}) &= \sum_{i=0}^3 (g^i)^* \otimes g^i \cdot p_{1,j} + \sum_{i=0}^3 (xg^i)^* \otimes xg^i \cdot p_{1,j} \\
&= \sum_{i=0}^3 (g^i)^* \otimes ((-1)^j)^i p_{1,j} + \sum_{i=0}^3 (xg^i)^* \otimes ((-1)^{j+1})^i (-1)^j (p_{2,j} + \sqrt{2}p_{3,j}) \\
&= (a^2)^j \otimes p_{1,j} - \frac{\xi\sqrt{2}}{2} (-a^2)^j ac \otimes (p_{2,j} + \sqrt{2}p_{3,j}); \\
\delta(p_{2,j}) &= \sum_{i=0}^3 (g^i)^* \otimes g^i \cdot p_{2,j} + \sum_{i=0}^3 (xg^i)^* \otimes xg^i \cdot p_{2,j} \\
&= \sum_{i=0}^3 (g^i)^* \otimes ((-1)^{j+1})^i p_{2,j} + \sum_{i=0}^3 (xg^i)^* \otimes (-\sqrt{2}(-1)^j ((-1)^j)^i) p_{4,j} \\
&= (a^2)^{j+1} \otimes p_{2,j} + \xi(-a^2)^j ab \otimes p_{4,j}; \\
\delta(p_{3,j}) &= \sum_{i=0}^3 (g^i)^* \otimes g^i \cdot p_{3,j} + \sum_{i=0}^3 (xg^i)^* \otimes xg^i \cdot p_{3,j} \\
&= \sum_{i=0}^3 (g^i)^* \otimes ((-1)^{j+1})^i p_{3,j} + \sum_{i=0}^3 (xg^i)^* \otimes (-1)^j ((-1)^j)^i p_{4,j} \\
&= (a^2)^{j+1} \otimes p_{3,j} - \frac{\xi\sqrt{2}}{2} (-a^2)^j ab \otimes p_{4,j}; \\
\delta(p_{4,j}) &= \sum_{i=0}^3 (g^i)^* \otimes g^i \cdot p_{4,j} + \sum_{i=0}^3 (xg^i)^* \otimes xg^i \cdot p_{4,j} = \sum_{i=0}^3 (g^i)^* \otimes ((-1)^j)^i p_{4,j} \\
&= (a^2)^j \otimes p_{4,j}.
\end{aligned}$$

□

Finally, we describe the braiding for every  $P_j \in {}^{\mathcal{K}}\mathcal{YD}$ ,  $j \in \mathbb{Z}_4$ . The proof follows by a straightforward computation using (1) and the coaction given in Proposition 3.6.

**Proposition 3.7.** *Let  $j \in \mathbb{Z}_4$ . The braiding of  $P_j \in {}^{\mathcal{K}}\mathcal{YD}$  is given by the formulae:*

$$\begin{aligned}
c(p_{1,j} \otimes \begin{pmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \end{pmatrix}) &= \begin{pmatrix} (-1)^j p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ (-1)^j p_{4,j} \end{pmatrix} \otimes p_{1,j} + \frac{\sqrt{2}}{2} \begin{pmatrix} -p_{3,j} \\ (-1)^j p_{4,j} \\ 0 \\ 0 \end{pmatrix} \otimes (p_{2,j} + \sqrt{2}p_{3,j}), \\
c(p_{2,j} \otimes \begin{pmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \end{pmatrix}) &= \begin{pmatrix} p_{1,j} \\ (-1)^{j+1} p_{2,j} \\ (-1)^{j+1} p_{3,j} \\ p_{4,j} \end{pmatrix} \otimes p_{2,j} + \begin{pmatrix} (-1)^{j+1} p_{3,j} \\ -p_{4,j} \\ 0 \\ 0 \end{pmatrix} \otimes p_{4,j}, \\
c(p_{3,j} \otimes \begin{pmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \end{pmatrix}) &= \begin{pmatrix} p_{1,j} \\ (-1)^{j+1} p_{2,j} \\ (-1)^{j+1} p_{3,j} \\ p_{4,j} \end{pmatrix} \otimes p_{3,j} + \frac{\sqrt{2}}{2} \begin{pmatrix} (-1)^j p_{3,j} \\ p_{4,j} \\ 0 \\ 0 \end{pmatrix} \otimes p_{4,j}, \\
c(p_{4,j} \otimes \begin{pmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \end{pmatrix}) &= \begin{pmatrix} (-1)^j p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ (-1)^j p_{4,j} \end{pmatrix} \otimes p_{4,j}.
\end{aligned}$$

□

4. NICHOLS ALGEBRAS IN  ${}^{\mathcal{K}}\mathcal{YD}$ 

In this section we determine all finite-dimensional Nichols algebras of simple modules over  $\mathcal{K}$ . They consist of exterior algebras of dimension 2 and 8-dimensional algebras with triangular braiding. Indeed, since all objects in  ${}^{\mathcal{K}}\mathcal{YD}$  can be described as objects in the category of Yetter-Drinfeld modules over the pointed Hopf algebra  $\mathcal{K}^* = \mathcal{A}'_4$ , by [U] it follows that the associated braiding is triangular. These 8-dimensional examples are new examples of finite-dimensional Nichols algebras. They are isomorphic to quantum linear spaces as algebras, but not as coalgebras since the braiding differs; in our case, the braiding is not diagonal. It remains an open question if they are twist equivalent and in such a case, in which category.

We begin by studying the Nichols algebras of the one-dimensional simple modules and their projective covers.

**Lemma 4.1.** *Let  $j \in \mathbb{Z}_4$ . The Nichols algebras  $\mathfrak{B}(\mathbb{k}_{\chi^j})$  associated to  $\mathbb{k}_{\chi^j} = \mathbb{k}x$  are:*

$$\mathfrak{B}(\mathbb{k}_{\chi^j}) = \begin{cases} \mathbb{k}[x], & \text{if } j = 0, 2, \\ \mathbb{k}[x]/(x^2) = \bigwedge \mathbb{k}_{\chi^j}, & \text{if } j = 1, 3. \end{cases}$$

*Proof.* Since by Proposition 3.2 we have that  $c = (-1)^j \tau$ , where  $\tau$  represents the usual flip, the result follows immediately.  $\square$

**Corollary 4.2.** *Let  $W = \mathbb{k}_{\chi^{j_1}} \oplus \cdots \oplus \mathbb{k}_{\chi^{j_t}}$  be a direct sum of one-dimensional modules with  $j_s \in \{1, 3\}$  for all  $1 \leq s \leq t$ . Then  $\mathfrak{B}(W) = \bigwedge W \simeq \mathfrak{B}(\mathbb{k}_{\chi^{j_1}}) \otimes \cdots \otimes \mathfrak{B}(\mathbb{k}_{\chi^{j_t}})$ .*

*Proof.* If  $j_s \in \{1, 3\}$  for all  $1 \leq s \leq t$ , then the braiding on  $W \otimes W$  is  $-\tau$  and hence  $\mathfrak{B}(W) = \bigwedge W$ . Let  $v \in \mathbb{k}_{\chi^{j_r}}$ ,  $w \in \mathbb{k}_{\chi^{j_s}}$ , then  $c(v \otimes w) = (a^2)^{j_r j_s} w \otimes v = (-1)^{j_r j_s} w \otimes v$ . Hence,  $c_W^2 = \text{id}_W$  and the last assertion follows from [Gr, Theorem 2.2].  $\square$

**Lemma 4.3.** *Let  $j \in \mathbb{Z}_4$  and  $P_j$  be the projective cover of  $\mathbb{k}_{\chi^j}$ . Then  $\dim \mathfrak{B}(P_j) = \infty$ .*

*Proof.* In all cases, the braiding on  $P_j \otimes P_j$  contains an eigenvector of eigenvalue 1 and the claim follows by Remark 1.2. Indeed, by Proposition 3.7 we have that  $c(p_{4,j} \otimes p_{4,j}) = p_{4,j} \otimes p_{4,j}$  if  $j = 0, 2$ , and  $c(p_{3,j} \otimes p_{3,j}) = p_{3,j} \otimes p_{3,j}$  if  $j = 1, 3$ .  $\square$

Before we describe the Nichols algebras associated to 2-dimensional simple modules, we analyze the Nichols algebras of non-simple indecomposable modules. It turns out that they are all infinite-dimensional.

**Remark 4.4.** *Let  $V \in {}^{\mathcal{K}}\mathcal{YD}$  be a finite-dimensional module such that  $\dim \mathfrak{B}(V) < \infty$ . Since taking the Nichols algebra defines a functor between the category of braided vector spaces and the category of braided Hopf algebras, see [AG], it follows that  $\dim \mathfrak{B}(W) < \infty$  for all  $W \in \text{Soc}(V)$  or  $W \in \text{Top}(V)$ . Furthermore, if  $0 \subseteq \text{Soc}(V) \subseteq \text{Soc}^2(V) \subseteq \cdots \subseteq \text{Soc}^n(V) = V$  denotes the socle series of  $V$ , then  $\dim \mathfrak{B}(V/\text{Soc}^i(V))$ ,  $\dim \mathfrak{B}(\text{Soc}^i(V))$  and  $\dim \mathfrak{B}(\text{Soc}(V/\text{Soc}^i(V)))$  are all finite for all  $1 \leq i \leq n$ .*

**Theorem 4.5.** *Let  $M \in {}^{\mathcal{K}}\mathcal{YD}$  be a finite-dimensional non-simple indecomposable module. Then  $\mathfrak{B}(M)$  is infinite-dimensional.*

*Proof.* Assume first that  $\dim M = 2$ . Then by Lemma 2.15 (i), we have that  $M \simeq M_\ell^+$  or  $M \simeq M_\ell^-$  for some  $0 \leq \ell \leq 3$ . Since  $\text{Soc}(M_\ell^+) = \mathbb{k}_{\chi^\ell}$ ,  $\text{Top}(M_\ell^+) = \mathbb{k}_{\chi^{\ell+1}}$  and  $\text{Soc}(M_\ell^-) = \mathbb{k}_{\chi^\ell}$ ,  $\text{Top}(M_\ell^-) = \mathbb{k}_{\chi^{\ell-1}}$ , it follows that  $\dim \mathfrak{B}(M) = \infty$  by Lemma 4.1 and Remark 4.4 because  $\dim \mathfrak{B}(N)$  is infinite for a simple module either in  $\text{Soc}(M)$  or in  $\text{Top}(M)$ .

Assume now that  $\dim M = n \geq 3$ . We prove the claim by induction on  $\dim M$ . Suppose that  $\dim \mathfrak{B}(N)$  is infinite for all indecomposable module of dimension less than  $n$ . By Remark 2.18,  $\text{Soc}(M)$  consists of one-dimensional modules. Let  $\bar{N}$  be a simple module contained in  $\text{Soc}(M/\text{Soc}(M))$  and denote by  $N$  the corresponding submodule of  $M$ . Then,  $\dim \bar{N} = 1$  by Remark 2.18. If  $\text{Soc}(M) = \mathbb{k}_\lambda$ , then  $N$  is an indecomposable module of dimension 2. The previous paragraphs imply that  $\dim \mathfrak{B}(N)$  is infinite and consequently  $\dim \mathfrak{B}(M)$  is infinite. Assume  $\text{Soc}(M)$  contains more than one simple module, and let



$\mathbb{k}_\lambda \subset \text{Soc}(M)$ . If  $N/\mathbb{k}_\lambda$  is semisimple, then  $N$  contains an indecomposable module of dimension 2 and whence  $\dim \mathfrak{B}(N/\mathbb{k}_\lambda)$  is infinite. This implies again that  $\dim \mathfrak{B}(N)$  and  $\dim \mathfrak{B}(M)$  are both infinite. If  $N/\mathbb{k}_\lambda$  is not semisimple, then it contains an indecomposable module of dimension less than  $n$ . By induction,  $\dim \mathfrak{B}(N/\mathbb{k}_\lambda)$  is infinite and the theorem follows applying the arguments above.  $\square$

**Remark 4.6.** Let  $V \in {}^{\mathcal{K}}\mathcal{YD}$  such that  $\dim \mathfrak{B}(V)$  is finite. Then by Theorem 4.5,  $V$  is necessarily semisimple. In these notes we will analyse only the Nichols algebras over simple modules, since the case of semisimple modules demands much more work to be carried out. A first approach could be done by studying the Yetter-Drinfeld submodules  $\text{ad}^n(V)(W)$  of a given Nichols algebra  $\mathfrak{B}(V \oplus W)$  with  $V$  and  $W$  simple modules, see [HS] for details. A direct computation shows that  $\dim \mathfrak{B}(V \oplus W) = \infty$  for  $V = \mathbb{k}_\chi$  and  $W = V_{3,1}, V_{3,3}$ ; and  $V = \mathbb{k}_{\chi^3}$  and  $W = V_{2,1}, V_{2,3}$ . In fact,  $\text{ad}(\mathbb{k}_\chi)(V_{3,1}) \simeq V_{0,3}$ ,  $\text{ad}(\mathbb{k}_\chi)(V_{3,3}) \simeq V_{0,1}$ ,  $\text{ad}(\mathbb{k}_{\chi^3})(V_{2,1}) \simeq V_{1,3}$  and  $\text{ad}(\mathbb{k}_{\chi^3})(V_{2,3}) \simeq V_{1,1}$ .

Now we proceed to analyze the Nichols algebras associated to two-dimensional simple modules. It turns out that 4 of these simple modules give rise to 8-dimensional Nichols algebras with triangular (and not diagonal) braiding.

**Lemma 4.7.** Let  $\Lambda' = \Lambda \setminus \{(2,1), (3,1), (2,3), (3,3)\}$ . Then  $\dim \mathfrak{B}(V_{i,j}) = \infty$  for all  $(i,j) \in \Lambda'$ .

*Proof.* In all cases, the braiding of  $V_{i,j}$  contains an eigenvector  $w \otimes w$  with  $w \in V_{i,j}$  of eigenvalue 1, hence the lemma follows by Remark 1.2. Indeed, if  $(i,j) = (1,1)$  or  $(1,3)$ , then  $w = e_1 + \sqrt{2}\xi e_2$ . Otherwise, take  $w = e_1$ .  $\square$

Next we describe the Nichols algebras associated to the pairs  $(2,1), (3,1), (2,3)$  and  $(3,3)$  in  $\Lambda$ . By Remark 2.8 we have that  $\mathfrak{B}(V_{2,1})^* \simeq \mathfrak{B}(V_{3,3})$  and  $\mathfrak{B}(V_{3,1})^* \simeq \mathfrak{B}(V_{2,3})$ . It turns out that, as algebras  $\mathfrak{B}(V_{2,1}) \simeq \mathfrak{B}(V_{2,3})$  and  $\mathfrak{B}(V_{3,1}) \simeq \mathfrak{B}(V_{3,3})$  and all of them are isomorphic to algebras associated to quantum linear spaces.

Remember that every graded Hopf algebra in  ${}^{\mathcal{K}}\mathcal{YD}$  satisfies the Poincaré duality [AG, Proposition 3.2.2], i.e. if  $R = \bigoplus_{i=0}^N R^i$  with  $N$  minimal such that  $R^N \neq \{0\}$ , then  $\dim R^i = \dim R^{N-i}$ .

**Proposition 4.8.**  $\mathfrak{B}(V_{2,1}) = \mathbb{k}\langle x, y : x^2 = 0, xy + \xi yx = 0, y^4 = 0 \rangle$ ; in particular  $\dim \mathfrak{B}(V_{2,1}) = 8$ .

*Proof.* Write  $x = e_1, y = e_2$  for the generators of  $V_{2,1}$ . Then, the coaction given by Proposition 3.3 is  $\delta(x) = d \otimes x + (-1 + \xi)c \otimes y$  and  $\delta(y) = a \otimes y + \frac{1}{2}(-1 - \xi)b \otimes x$ . Using the braiding given by Proposition 3.4, we get that

$$c\left(\begin{Bmatrix} x \\ y \end{Bmatrix}\right) \otimes \left\{ \begin{array}{cc} x & y \end{array} \right\} = \left\{ \begin{array}{cc} -x \otimes x & -\xi y \otimes x + (\xi - 1)x \otimes y \\ -x \otimes y & \xi y \otimes y - \frac{1}{2}(1 + \xi)x \otimes x \end{array} \right\},$$

and one sees that the relations  $x^2 = 0, xy + \xi yx = 0$  and  $y^4 = 0$  must hold in  $\mathfrak{B}(V_{2,1})$ . Indeed, the first two relations are easily checked since these elements are primitive. Let us focus in the last one; we show that it is also primitive. Since

$$\begin{aligned} \Delta(y^2) &= (y \otimes 1 + 1 \otimes y)(y \otimes 1 + 1 \otimes y) \\ &= y^2 \otimes 1 + (1 + \xi)y \otimes y - \frac{1}{2}(1 + \xi)x \otimes x + 1 \otimes y^2, \end{aligned}$$

then

$$\begin{aligned} \Delta(y^3) &= (y \otimes 1 + 1 \otimes y)(y^2 \otimes 1 + (1 + \xi)y \otimes y - \frac{1}{2}(1 + \xi)x \otimes x + 1 \otimes y^2) \\ &= y^3 \otimes 1 + \frac{1}{2}(1 + \xi)xy \otimes x + \xi y^2 \otimes y - \frac{1}{2}(1 + \xi)x \otimes xy + \xi y \otimes y^2 + 1 \otimes y^3 \end{aligned}$$

because  $c(y \otimes y^2) = (x^2 - y^2) \otimes y + \frac{1}{2}(1 - \xi)(yx - xy) \otimes x$ , and

$$\begin{aligned} \Delta(y^4) &= (y \otimes 1 + 1 \otimes y)(y^3 \otimes 1 + \frac{1}{2}(1 + \xi)xy \otimes x + \xi y^2 \otimes y - \\ &\quad - \frac{1}{2}(1 + \xi)x \otimes xy + \xi y \otimes y^2 + 1 \otimes y^3) \\ &= y^4 \otimes 1 + 1 \otimes y^4, \end{aligned}$$

because  $c(y \otimes y^3) = \xi(-y^3 + x^2y + yx^2 - xyx) \otimes y + \frac{1}{2}(1 + \xi)(y^2x - x^3 - yxy + xy^2) \otimes x$  and  $c(y \otimes xy) = -\xi xy \otimes y + \frac{1}{2}(1 + \xi)x^2 \otimes x$ .

Therefore, we have a graded braided Hopf algebra epimorphism  $\pi : T(V_{2,1})/I \twoheadrightarrow \mathfrak{B}(V_{2,1})$  where  $I$  is the ideal generated by the relations. Let  $R = T(V_{2,1})/I$ , then  $R$  is a graded braided Hopf algebra with  $R^5 = 0$ ,  $R^0 = \mathbb{k}$  and  $R^1 = V_{2,1}$ . Since  $R$  satisfies the Poincaré duality, we have that  $\dim R^4 = 1$  and  $\dim R^3 = 2$ . Clearly,  $\dim R^2 = 2$ . As  $\mathfrak{B}^5(V_{2,1}) = 0$  and  $\pi$  is injective in degree 0 and 1, it follows that  $\dim \mathfrak{B}^4(V_{2,1}) = \dim \mathfrak{B}^0(V_{2,1}) = 1$  and  $\dim \mathfrak{B}^3(V_{2,1}) = 2 = \dim \mathfrak{B}^1(V_{2,1})$ . Then  $\pi$  is injective in degree 4 and 3 also. In order to prove that  $\pi$  is injective, it remains to show that  $\pi$  is injective in degree 2. This is equivalent to check that the relations in degree 2 in the Nichols algebra are just  $x^2 = 0, xy + \xi yx = 0$ , which follows by a direct computation using the braiding.  $\square$

**Proposition 4.9.**  $\mathfrak{B}(V_{2,3}) = \mathbb{k}\langle x, y : x^2 = 0, xy - \xi yx = 0, y^4 = 0 \rangle$ ; in particular,  $\dim \mathfrak{B}(V_{2,3}) = 8$ .

*Proof.* The proof follows the same lines of Proposition 4.8. We only show that the relations hold. Write  $x = e_1, y = e_2$  for the generators of  $V_{3,1}$ , then the coaction given by Proposition 3.3 is  $\delta(x) = a \otimes x + (-1 - \xi)b \otimes y$  and  $\delta(y) = d \otimes y + \frac{1}{2}(-1 + \xi)c \otimes x$ . Using the braiding, we see that

$$\begin{aligned} \Delta(x^2) &= x^2 \otimes 1 + 1 \otimes x^2, \\ \Delta(xy) &= xy \otimes 1 - \xi x \otimes y + \xi y \otimes x + 1 \otimes xy, \\ \Delta(yx) &= yx \otimes 1 + y \otimes x - x \otimes y + 1 \otimes yx, \\ \Delta(y^2) &= y^2 \otimes 1 + (1 - \xi)y \otimes y + \frac{1}{2}(\xi - 1)x \otimes x + 1 \otimes y^2, \end{aligned}$$

which gives us the relations  $x^2 = 0$  and  $xy - \xi yx = 0$ . Since  $c(y \otimes y^2) = (x^2 - y^2) \otimes y + \frac{1}{2}(1 + \xi)(yx - xy) \otimes x$ , we have that

$$\Delta(y^3) = y^3 \otimes 1 + \frac{1}{2}(1 - \xi)xy \otimes x - \xi y^2 \otimes y + \frac{1}{2}(\xi - 1)x \otimes xy - \xi y \otimes y^2 + 1 \otimes y^3,$$

and whence

$$\begin{aligned} \Delta(y^4) &= \Delta(y^3)(y \otimes 1 + 1 \otimes y) \\ &= (y^3 \otimes 1 + \frac{1}{2}(1 - \xi)xy \otimes x - \xi y^2 \otimes y + \frac{1}{2}(\xi - 1)x \otimes xy - \xi y \otimes y^2 + 1 \otimes y^3) \\ &\quad (y \otimes 1 + 1 \otimes y), \\ &= y^4 \otimes 1 + 1 \otimes y^4, \end{aligned}$$

since  $c(y \otimes y^3) = -\xi(-y^3 + x^2y + yx^2 - xyx) \otimes y + \frac{1}{2}(1 - \xi)(y^2x - x^3 - yxy + xy^2) \otimes x$  and  $c(y \otimes xy) = \xi xy \otimes y + \frac{1}{2}(1 - \xi)x^2 \otimes x$ . Hence, we have the relation  $y^4 = 0$ .  $\square$

**Proposition 4.10.**  $\mathfrak{B}(V_{3,1}) = \mathbb{k}\langle x, y : x^2 - 2y^2 = 0, xy + yx = 0, y^4 = 0 \rangle$ ; in particular  $\dim \mathfrak{B}(V_{3,1}) = 8$ .

*Proof.* The proof follows the same lines of Proposition 4.8. Again, we only show that the relations hold. Write  $x = e_1$ ,  $y = e_2$  for the generators of  $V_{3,1}$ . First, note that the coaction given by Proposition 3.3 is  $\delta(x) = d \otimes x + (\xi - 1)c \otimes y$  and  $\delta(y) = a \otimes y + \frac{1}{2}(-\xi - 1)b \otimes x$ . Then, using the action of  $K$  and the braiding given by Proposition 3.4, we get that

$$\begin{aligned}\Delta(x^2) &= x^2 \otimes 1 + (1 + \xi)x \otimes x + 1 \otimes x^2, & \Delta(xy) &= xy \otimes 1 + \xi x \otimes y - y \otimes x + 1 \otimes xy, \\ \Delta(yx) &= yx \otimes 1 + y \otimes x - \xi x \otimes y + 1 \otimes yx, & \Delta(y^2) &= y^2 \otimes 1 + \frac{1}{2}(1 + \xi)x \otimes x + 1 \otimes y^2,\end{aligned}$$

which gives us the relations  $x^2 - 2y^2 = 0$  and  $xy + yx = 0$ , since both elements are primitive. Analogously, since  $c(y \otimes y^2) = (y^2 - x^2) \otimes y - \frac{1}{2}(1 + \xi)(xy + yx) \otimes x$ , we have that

$$\Delta(y^3) = y^3 \otimes 1 - \frac{1}{2}(1 + \xi)xy \otimes x - y^2 \otimes y - \frac{1}{2}(1 - \xi)x \otimes xy + y \otimes y^2 + 1 \otimes y^3,$$

and consequently

$$\begin{aligned}\Delta(y^4) &= \Delta(y^3)(y \otimes 1 + 1 \otimes y) = \\ &= (y^3 \otimes 1 - \frac{1}{2}(1 + \xi)xy \otimes x - y^2 \otimes y - \frac{1}{2}(1 - \xi)x \otimes xy + y \otimes y^2 + 1 \otimes y^3)(y \otimes 1 + 1 \otimes y) \\ &= y^4 \otimes 1 + 1 \otimes y^4,\end{aligned}$$

since  $c(y \otimes y^3) = (yx^2 + xyx - y^3 + x^2y) \otimes y + \frac{1}{2}(1 + \xi)(yxy - x^3 + xy^2 + y^2x) \otimes x$  and  $c(y \otimes xy) = \xi xy \otimes y + \frac{1}{2}(1 - \xi)x^2 \otimes x$ . Thereby, we have the relation  $y^4 = 0$ .  $\square$

**Proposition 4.11.**  $\mathfrak{B}(V_{3,3}) = \mathbb{k}\langle x, y : x^2 - 2y^2 = 0, xy + yx = 0, y^4 = 0 \rangle$ ; in particular,  $\dim \mathfrak{B}(V_{3,3}) = 8$ .

*Proof.* As in the proof of Proposition 4.10, we only show that the relations hold. Write  $x = e_1$ ,  $y = e_2$  for the generators of  $V_{3,3}$ , then the coaction given by Proposition 3.3 is  $\delta(x) = a \otimes x + (\xi + 1)b \otimes y$  and  $\delta(y) = d \otimes y + \frac{1}{2}(1 - \xi)c \otimes x$ . Using the braiding given in Proposition 3.4, we have

$$\begin{aligned}\Delta(x^2) &= x^2 \otimes 1 + (1 - \xi)x \otimes x + 1 \otimes x^2, & \Delta(xy) &= xy \otimes 1 - \xi x \otimes y - y \otimes x + 1 \otimes xy, \\ \Delta(yx) &= yx \otimes 1 + y \otimes x + \xi x \otimes y + 1 \otimes yx, & \Delta(y^2) &= y^2 \otimes 1 + \frac{1}{2}(1 - \xi)x \otimes x + 1 \otimes y^2,\end{aligned}$$

which gives us the relations  $x^2 - 2y^2 = 0$  and  $xy + yx = 0$ . Since  $c(y \otimes y^2) = (y^2 - x^2) \otimes y + \frac{1}{2}(\xi - 1)(xy + yx) \otimes x$ , it follows that

$$\Delta(y^3) = y^3 \otimes 1 + \frac{1}{2}(\xi - 1)xy \otimes x - y^2 \otimes y - \frac{1}{2}(1 + \xi)x \otimes xy + y \otimes y^2 + 1 \otimes y^3,$$

and consequently

$$\begin{aligned}\Delta(y^4) &= \Delta(y^3)(y \otimes 1 + 1 \otimes y) = \\ &= (y^3 \otimes 1 + \frac{1}{2}(\xi - 1)xy \otimes x - y^2 \otimes y - \frac{1}{2}(1 + \xi)x \otimes xy + y \otimes y^2 + 1 \otimes y^3)(y \otimes 1 + 1 \otimes y) \\ &= y^4 \otimes 1 + 1 \otimes y^4,\end{aligned}$$

since  $c(y \otimes y^3) = (yx^2 + xyx - y^3 + x^2y) \otimes y + \frac{1}{2}(1 - \xi)(yxy - x^3 + xy^2 + y^2x) \otimes x$  and  $c(y \otimes xy) = -\xi xy \otimes y + \frac{1}{2}(1 + \xi)x^2 \otimes x$ . Thus,  $y^4 = 0$  since it is also a primitive element.  $\square$

We end this section with the characterization of the finite-dimensional Nichols algebras over indecomposable objects in  ${}^{\mathcal{K}}\mathcal{YD}$ .

**Proof of Theorem A.** Let  $V$  be an indecomposable module such that  $\mathfrak{B}(V)$  is finite-dimensional. Then by Theorem 4.5,  $V$  is necessarily simple. The claim then follows by Lemmata 4.1 and 4.7, and Propositions 4.8, 4.9, 4.10 and 4.11. Clearly, Nichols algebras over distinct families are pairwise non-isomorphic, since they are generated by the set of primitive elements which are non-isomorphic as Yetter–Drinfeld modules.  $\square$

## 5. HOPF ALGEBRAS OVER $\mathcal{K}$

In this last section we determine all finite-dimensional Hopf algebras  $H$  such that  $H_{[0]} = \mathcal{K}$  and the corresponding infinitesimal braiding is a simple module in  ${}^{\mathcal{K}}\mathcal{YD}$ . That is, the graded algebra with respect to the standard filtration is  $\text{gr } H = \bigoplus_{i \geq 0} H_{[i]}/H_{[i-1]} \simeq R \# \mathcal{K}$  with  $R$  a connected graded braided Hopf algebra in  ${}^{\mathcal{K}}\mathcal{YD}$  and  $R^1$  isomorphic to a simple object in  ${}^{\mathcal{K}}\mathcal{YD}$ .

We begin by proving that this type of Hopf algebra is generated in degree one, *i.e.*  $R \simeq \mathfrak{B}(V)$  with  $V = R^1$ .

**Theorem 5.1.** *Let  $H$  be a finite-dimensional Hopf algebra over  $\mathcal{K}$  such that its infinitesimal braiding is isomorphic to a simple object in  ${}^{\mathcal{K}}\mathcal{YD}$ . Then the diagram of  $H$  is a Nichols algebra, and consequently  $H$  is generated by the elements of degree one with respect to the standard filtration.*

*Proof.* As  $H$  is a Hopf algebra over  $\mathcal{K}$ , we have that  $H_{[0]} \simeq \mathcal{K}$ ,  $\text{gr } H = R \# \mathcal{K}$ , and by hypothesis  $R^1 = V$ , with  $V$  a simple object in  ${}^{\mathcal{K}}\mathcal{YD}$ . Let  $S = R^*$  be the graded dual of the diagram  $R$ . By [AS1, Lemma 5.5],  $S$  is generated by  $W = S(1)$  and  $R$  is a Nichols algebra if and only if  $P(S) = S(1)$ , that is, if  $S$  is also a Nichols algebra. Since  $\mathfrak{B}(W) = T(W)/J$ , to show that  $P(S) = S(1)$ , it is enough to prove that the relations that generate the ideal  $J$  hold in  $S$ . This can be done by a case by case computation. We perform here two cases, leaving the rest as exercise for the reader.

Assume  $W = \mathbb{k}_{\chi^\ell}$  with  $1 \leq \ell \leq 3$ . Then by Lemma 4.1,  $\ell = 1, 3$  and  $\mathfrak{B}(W) \simeq \bigwedge W = \mathbb{k}[x]/(x^2)$ . If we denote  $r = x^2 \in S$ , then  $r$  is primitive and  $c(r \otimes r) = r \otimes r$ , where  $c$  is the braiding in  ${}^{\mathcal{K}}\mathcal{YD}$ . Since  $S$  is finite-dimensional, it follows that  $r = 0$  and there is a surjection  $\mathfrak{B}(W) \twoheadrightarrow S$ . In particular,  $P(S) = W = S(1)$  and  $S$  is a Nichols algebra.

Assume  $\dim W = 2$ , then by Theorem A we have that  $W \simeq V_{2,1}, V_{2,3}, V_{3,1}$  or  $V_{3,3}$ . We prove the statement only for  $W \simeq V_{2,1}$ . By Proposition 4.8,  $V_{2,1}$  is generated by the elements  $x, y$  and the ideal defining the Nichols algebra is generated by the elements  $x^2, xy + \xi yx$  and  $y^4$ . Moreover, by the proof of the same proposition, these elements are primitive. Thus, we need to prove that  $c(r \otimes r) = r \otimes r$  for  $r = x^2, xy + \xi yx$  and  $y^4$ . Using the coaction given by Proposition 3.3, we have that  $\delta(x^2) = d^2 \otimes x^2 + (\xi - 1)dc \otimes (xy + \xi yx)$  and whence

$$\begin{aligned} c(x^2 \otimes x^2) &= (x^2)_{(-1)} \cdot x^2 \otimes (x^2)_{(0)} = d^2 \cdot x^2 \otimes x^2 + (\xi - 1)(dc) \cdot x^2 \otimes (xy + \xi yx) \\ &= (-1)^2 x^2 \otimes x^2 = x^2 \otimes x^2. \end{aligned}$$

Analogously, since  $\delta(xy) = 1 \otimes xy + (\xi - 1)ca \otimes y^2$  and  $\delta(yx) = 1 \otimes yx + (\xi - 1)ac \otimes y^2$ , one has that  $\delta(xy + \xi yx) = 1 \otimes (xy + \xi yx)$ . This implies that  $c((xy + \xi yx) \otimes (xy + \xi yx)) = (xy + \xi yx) \otimes (xy + \xi yx)$ .

Finally, we have that  $\delta(y^2) = a^2 \otimes y^2 - \frac{1}{2}(1 + \xi)ba \otimes (xy + \xi yx) = a^2 \otimes y^2$ . Thus,  $\delta(y^4) = 1 \otimes y^4$  and consequently  $c(y^4 \otimes y^4) = y^4 \otimes y^4$ .  $\square$

Recall that, if  $v \in V = R^1$  is a primitive element, then by the formula given by the bosonization we have that  $\Delta(v \# 1) = v^{(1)} \# (v^{(2)})_{(-1)} \otimes (v^{(2)})_{(0)} \# 1 \in H_{[1]} \otimes H_{[0]} + H_{[0]} \otimes H_{[1]}$ . We denote  $v = v \# 1$  for all  $v \in V$ , and  $k = 1 \# k$  for all  $k \in \mathcal{K}$ .

Next we show that the bosonizations of the Nichols algebras associated to the simple modules  $\mathbb{k}_{\chi^\ell}$  with  $\ell = 1, 3$  and  $V_{2,1}, V_{2,3}$  do not admit deformations.

**Proposition 5.2.** *Let  $H$  be a finite-dimensional Hopf algebra over  $\mathcal{K}$  such that its infinitesimal braiding  $V$  is isomorphic either to  $\mathbb{k}_{\chi^\ell}$  with  $\ell = 1$  or  $3$ . Then  $H \simeq \bigwedge \mathbb{k}_{\chi^\ell} \# \mathcal{K}$ .*

*Proof.* By Theorem 5.1, we know that  $\text{gr } H = \bigoplus_{i \geq 0} H_{[i]}/H_{[i-1]} \simeq \bigwedge \mathbb{k}_{\chi^\ell} \# \mathcal{K}$  with  $\ell = 1$  or  $3$ . We prove that the relations also hold in  $H$ . Since the relations are homogeneous, it follows that  $H \simeq \text{gr } H \simeq \bigwedge \mathbb{k}_{\chi^\ell} \# \mathcal{K}$ . Write  $\bigwedge \mathbb{k}_{\chi^\ell} = \mathbb{k}[x]/(x^2)$ . Then  $\delta(x) = a^2 \otimes x$  and consequently

$$\Delta(x^2) = x^2 \otimes 1 + 1 \otimes x^2 + (a^2 \cdot x + x) \otimes x = x^2 \otimes 1 + 1 \otimes x^2.$$

Since  $\mathcal{K}$  does not contain primitive elements, it follows that the relation  $x^2 = 0$  must hold also in  $H$ .  $\square$

**Proposition 5.3.** *Let  $H$  be a finite-dimensional Hopf algebra over  $\mathcal{K}$  such that its infinitesimal braiding  $V$  is isomorphic either to  $V_{2,1}$  or  $V_{2,3}$ . Then  $H \simeq \mathfrak{B}(V) \# \mathcal{K}$ .*

*Proof.* We know that  $\text{gr } H = \bigoplus_{i \geq 0} H_{[i]}/H_{[i-1]} \simeq \mathfrak{B}(V) \# \mathcal{K}$  with  $V$  isomorphic either to  $V_{2,1}$  or  $V_{2,3}$ . As before, we prove that the homogeneous relations also hold in  $H$ .

Assume first that  $V \simeq V_{2,1}$ . Recall that  $\mathfrak{B}(V_{2,1}) \# \mathcal{K}$  is the algebra generated by  $x, y, a, b, c, d$  with  $x, y$  satisfying the the relations of  $\mathfrak{B}(V_{2,1})$ , see Proposition 4.8,  $a, b, c, d$  satisfying the the relations of  $\mathcal{K}$ , and all together satisfying the relations that give the commutativity:

$$(5) \quad \begin{aligned} ax &= -xa, & ay &= \xi ya + xc, & bx &= -xb, & by &= \xi yb + xd, \\ cx &= -xc, & cy &= -\xi yc + xa, & dx &= -xd, & dy &= -\xi yd + xb. \end{aligned}$$

As

$$\Delta(x) = x \otimes 1 + d \otimes x + (\xi - 1)c \otimes y \quad \text{and} \quad \Delta(y) = y \otimes 1 + a \otimes y - \frac{\xi + 1}{2}b \otimes x,$$

we have that

$$\Delta(xy + \xi yx) = (xy + \xi yx) \otimes 1 + 1 \otimes (xy + \xi yx) \quad \text{and} \quad \Delta(y^4) = y^4 \otimes 1 + 1 \otimes y^4.$$

Since  $\mathcal{K}$  does not contain primitive elements, it follows that the relations  $xy + \xi yx = 0$  and  $y^4 = 0$  hold in  $H$ . On the other hand,  $\Delta(x^2) = x^2 \otimes 1 + a^2 \otimes x^2 + (\xi - 1)ab \otimes (xy + \xi yx) = x^2 \otimes 1 + a^2 \otimes x^2$ , which implies that  $x^2$  is a  $(1, a^2)$ -primitive element in  $H_{[1]}$ . Since  $P_{1,a^2}(H_{[1]}) = P_{1,a^2}(\mathcal{K}) = \mathbb{k}\{1 - a^2, ab\}$ , we must have that

$$x^2 = \mu_1(1 - a^2) + \mu_2 ab \quad \text{for some } \mu_1, \mu_2 \in \mathbb{k}.$$

As  $ax^2 = x^2a$  and  $bx^2 = x^2b$ , but

$$\begin{aligned} a(\mu_1(1 - a^2) + \mu_2 ab) - (\mu_1(1 - a^2) + \mu_2 ab)a &= \mu_2(1 + \xi)c, \\ b(\mu_1(1 - a^2) + \mu_2 ab) - (\mu_1(1 - a^2) + \mu_2 ab)b &= 2\mu_1 c, \end{aligned}$$

it follows that  $\mu_1 = \mu_2 = 0$ . Therefore, the relation  $x^2 = 0$  also holds in  $H$  and consequently,  $H \simeq \text{gr } H$ .

Assume now  $V \simeq V_{2,3}$  and recall that  $\mathfrak{B}(V_{2,3}) \# \mathcal{K} = \mathfrak{B}(V_{2,1}) \# \mathcal{K}$  as algebras. Since

$$\Delta(x) = x \otimes 1 + a \otimes x - (1 + \xi)b \otimes y \quad \text{and} \quad \Delta(y) = y \otimes 1 + d \otimes y + \frac{\xi - 1}{2}c \otimes x,$$

we have that

$$\Delta(xy - \xi yx) = (xy - \xi yx) \otimes 1 + 1 \otimes (xy - \xi yx) \quad \text{and} \quad \Delta(y^4) = y^4 \otimes 1 + 1 \otimes y^4.$$

Again, since  $\mathcal{K}$  does not contain primitive elements, it follows that the relations  $xy - \xi yx = 0$  and  $y^4 = 0$  hold in  $H$ . On the other hand, we have that

$$\Delta(x^2) = x^2 \otimes 1 + a^2 \otimes x^2 - (1 + \xi)ab \otimes (xy - \xi yx) = x^2 \otimes 1 + a^2 \otimes x^2.$$

In particular,  $x^2 \in P_{1,a^2}(H_{[1]})$ . Since  $P_{1,a^2}(H_{[1]}) = P_{1,a^2}(\mathcal{K}) = \mathbb{k}\{1 - a^2, ab\}$ , it follows that

$$x^2 = \mu_1(1 - a^2) + \mu_2 ab \quad \text{for some } \mu_1, \mu_2 \in \mathbb{k}.$$

Since  $ax^2 = x^2a$  and  $bx^2 = x^2b$ , but

$$\begin{aligned} a(\mu_1(1 - a^2) + \mu_2 ab) - (\mu_1(1 - a^2) + \mu_2 ab)a &= \mu_2(1 + \xi)c \\ b(\mu_1(1 - a^2) + \mu_2 ab) - (\mu_1(1 - a^2) + \mu_2 ab)b &= 2\mu_1 c \end{aligned}$$

we must have that  $\mu_1 = \mu_2 = 0$  and therefore  $H \simeq \text{gr } H$ .  $\square$

In the following we define two Hopf algebras  $\mathfrak{A}_{3,1}(\mu)$  and  $\mathfrak{A}_{3,3}(\mu)$  which are constructed by deforming the relations on the Nichols algebras  $\mathfrak{B}(V_{3,1})$  and  $\mathfrak{B}(V_{3,3})$  over  $\mathcal{K}$ , respectively. We will show that they are indeed liftings of bosonizations.

**Definition 5.4.** For  $\mu \in \mathbb{k}$ , denote by  $\mathfrak{A}_{3,1}(\mu)$  the algebra generated by the elements  $x, y, a, b, c, d$  satisfying the relations

$$\begin{aligned} ab &= \xi ba, & ac &= \xi ca, & 0 &= cb = bc, & cd &= \xi dc, & bd &= \xi db, \\ ad &= da, & ad &= 1, & 0 &= b^2 = c^2, & a^2c &= b, & a^4 &= 1, \\ ax &= -\xi xa, & ay &= -ya - xc, & bx &= -\xi xb, & by &= -yb - xd \\ cx &= \xi xc, & cy &= -yc + xa, & dx &= \xi xd, & dy &= -yd + xb, \end{aligned}$$

$$x^2 - 2y^2 = \mu(1 - a^2), \quad xy + yx = \xi\mu ac, \quad y^4 = -\mu y^2(1 - a^2) - \frac{\mu^2}{2}(1 - a^2).$$

It is a Hopf algebra with its structure determined by

$$\begin{aligned} \Delta(x) &= x \otimes 1 + d \otimes x + (\xi - 1)c \otimes y & \Delta(y) &= y \otimes 1 + a \otimes y + \frac{1}{2}(-\xi - 1)b \otimes x, \\ \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, \\ \varepsilon(a) &= \varepsilon(d) = 1, & \varepsilon(b) &= \varepsilon(c) = \varepsilon(x) = \varepsilon(y) = 0. \end{aligned}$$

In particular, the antipode is given by

$$\begin{aligned} \mathcal{S}(x) &= -ax - (1 + \xi)cy, & \mathcal{S}(y) &= -dy + \frac{1}{2}(\xi - 1)bx, & \mathcal{S}(a) &= d, \\ \mathcal{S}(b) &= \xi b, & \mathcal{S}(c) &= -\xi c, & \mathcal{S}(d) &= a. \end{aligned}$$

**Remark 5.5.** Clearly,  $\mathfrak{A}_{3,1}(0) \simeq \mathfrak{B}(V_{3,1}) \# \mathcal{K}$ . Also note that  $\mathfrak{A}_{3,1}(\mu)$  is the quotient of the algebra  $T(V_{3,1}) \otimes \mathcal{K}$  by the two-sided ideal generated by the elements given by the three last rows of equations. In particular, we have that

$$\begin{aligned} \Delta(ab) &= ab \otimes 1 + a^2 \otimes ab, & \Delta(ac) &= ac \otimes a^2 + 1 \otimes ac, & \Delta(a^2) &= a^2 \otimes a^2, \\ \mathcal{S}(ab) &= -ac, & \mathcal{S}(ac) &= ab, & \mathcal{S}(a^2) &= a^2. \end{aligned}$$

**Definition 5.6.** For  $\mu \in \mathbb{k}$ , denote by  $\mathfrak{A}_{3,3}(\mu)$  the Hopf algebra defined by  $\mathfrak{A}_{3,1}(\mu) = \mathfrak{A}_{3,3}(\mu)$  as algebra but as coalgebra

$$\begin{aligned} \Delta(x) &= x \otimes 1 + a \otimes x + (\xi + 1)b \otimes y, & \varepsilon(x) &= 0, \\ \Delta(y) &= y \otimes 1 + d \otimes y + \frac{1}{2}(1 - \xi)c \otimes x, & \varepsilon(y) &= 0, \end{aligned}$$

and the same counit and comultiplication for the elements  $a, b, c, d$  in  $\mathcal{K}$ . In particular, we have

$$\mathcal{S}(x) = -dx - (\xi - 1)by, \quad \mathcal{S}(y) = -ay + \frac{1}{2}(1 + \xi)cx.$$

**Remark 5.7.** As before,  $\mathfrak{A}_{3,3}(0) \simeq \mathfrak{B}(V_{3,3}) \# \mathcal{K}$  and  $\mathfrak{A}_{3,3}(\mu)$  is the quotient of the algebra  $T(V_{3,3}) \otimes \mathcal{K}$  by the two-sided ideal generated by the elements given by the three last rows of equations.

In the next lemma we show that the algebras  $\mathfrak{A}_{3,1}(\mu)$  and  $\mathfrak{A}_{3,3}(\mu)$  are finite-dimensional Hopf algebras over  $\mathcal{K}$ .

**Lemma 5.8.**  $\mathfrak{A}_{3,1}(\mu)$  and  $\mathfrak{A}_{3,3}(\mu)$  are finite-dimensional for all  $\mu \in \mathbb{k}$  and  $(\mathfrak{A}_{3,1}(\mu))_{[0]} \simeq \mathcal{K} \simeq (\mathfrak{A}_{3,3}(\mu))_{[0]}$ .



*Proof.* We prove the assertion for  $\mathfrak{A}_{3,1}(\mu)$ , being the proof for  $\mathfrak{A}_{3,3}(\mu)$  completely analogous. Let  $J_{3,1}$  be the two-sided ideal generated by the elements given by the three last rows of equations. Then  $\mathfrak{A}_{3,1}(\mu) = T(V_{3,1}) \otimes \mathcal{K} / J_{3,1}$ . Note that  $T(V_{3,1}) \otimes \mathcal{K}$  is a graded algebra with the gradation defined by the usual on  $T(V_{3,1})$  and all the elements in  $\mathcal{K}$  to be of degree 0.

Denote by  $\mathfrak{A}_0$  the subalgebra generated by the coalgebra  $C$  linearly spanned by  $a, b, c, d$ , then  $\mathfrak{A}_0$  is a Hopf subalgebra of  $\mathfrak{A}_{3,1}(\mu)$  which is isomorphic to  $\mathcal{K}$ . Indeed, consider the Hopf algebra map  $\varphi : \mathcal{K} \rightarrow \mathfrak{A}_{3,1}(\mu)$  given by the composition  $\mathcal{K} \hookrightarrow T(V_{3,1}) \otimes \mathcal{K} \twoheadrightarrow T(V_{3,1}) \otimes \mathcal{K} / J_{3,1}$ ; in particular,  $\mathfrak{A}_0 \simeq \varphi(\mathcal{K})$ . Since  $\dim \mathcal{K} = 8$ , to prove that  $\varphi(\mathcal{K}) \simeq \mathcal{K}$  it is enough to show that  $\dim \varphi(\mathcal{K}) > 4$ , since it is a divisor of 8 by the Nichols-Zoeller Theorem. This follows from the fact that  $\varphi(C) \simeq C$  as coalgebras, since with respect to the grading in  $T(V_{3,1}) \otimes \mathcal{K}$ , the relations in  $J_{3,1}$  do not involve relations only in degree 0. Since the elements  $a, b, c, d$  are linearly independent in  $\mathcal{K}$ , they are also *l.i.* in  $\mathfrak{A}_{3,1}(\mu)$ .

If we set  $\mathfrak{A}_1 = \mathcal{K}\{x, y\}$ ,  $\mathfrak{A}_2 = \mathfrak{A}_1 + \mathcal{K}\{xy, y^2\}$ ,  $\mathfrak{A}_3 = \mathfrak{A}_2 + \mathcal{K}\{xy^2, y^3\}$  and  $\mathfrak{A}_4 = \mathfrak{A}_3 + \mathcal{K}\{xy^3\}$ , we have that  $\{\mathfrak{A}_n\}_{0 \leq n \leq 3}$  is a coalgebra filtration of  $\mathfrak{A}_{3,1}(\mu)$ . In particular,  $(\mathfrak{A}_{3,1}(\mu))_0 \subseteq \mathcal{K}$  and consequently  $(\mathfrak{A}_{3,3}(\mu))_{[0]} = \mathcal{K}$ ; that is,  $\mathfrak{A}_{3,1}(\mu)$  is a Hopf algebra over  $\mathcal{K}$ . Hence,  $\mathfrak{A}_{3,1}(\mu)$  is a finite-dimensional Hopf algebra which is free over  $\mathcal{K}$ . In particular, 8 divides  $\dim \mathfrak{A}_{3,1}(\mu)$ . Besides,  $\mathfrak{A}_{3,1}$  is a  $\mathcal{K}$ -module with a set of generators  $\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}$ . Thus,  $\dim \mathfrak{A}_{3,1}(\mu) \leq 8 \dim \mathcal{K} = 64$ .  $\square$

From the proof of the lemma above it follows that  $\dim \mathfrak{A}_{3,1}(\mu), \dim \mathfrak{A}_{3,3}(\mu) \leq 8 \dim \mathcal{K} = 64$ . In the next lemma we show that the algebras  $\mathfrak{A}_{3,1}(\mu)$  and  $\mathfrak{A}_{3,3}(\mu)$  are liftings of  $\mathfrak{B}(V_{3,1}) \# \mathcal{K}$  and  $\mathfrak{B}(V_{3,3}) \# \mathcal{K}$  for all  $\mu \in \mathbb{k}$ , respectively.

**Lemma 5.9.**  *$\text{gr } \mathfrak{A}_{3,1}(\mu) \simeq \mathfrak{B}(V_{3,1}) \# \mathcal{K}$  and  $\text{gr } \mathfrak{A}_{3,3}(\mu) \simeq \mathfrak{B}(V_{3,3}) \# \mathcal{K}$ .*

*Proof.* To prove the lemma, it is enough to show that  $\dim \mathfrak{A}_{3,1}(\mu), \dim \mathfrak{A}_{3,3}(\mu) \geq 64$ , since by the proof of Lemma 5.8 we have that  $\text{gr } \mathfrak{A}_{3,1}(\mu) \simeq R_{3,1} \# \mathcal{K}$  and  $\text{gr } \mathfrak{A}_{3,3}(\mu) \simeq R_{3,3} \# \mathcal{K}$  where  $R_{3,1}, R_{3,3}$  are  $K$ -modules linearly spanned by the set  $\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}$ . We show that the set  $B = \{x^i y^j a^k b^l : 0 \leq i, l \leq 1, 0 \leq j, k \leq 3\}$  is linearly independent by using adequate representations. As  $\mathfrak{A}_{3,1}(\mu)$  and  $\mathfrak{A}_{3,3}(\mu)$  have the same algebra structure, we prove it only for  $\mathfrak{A}_{3,1}(\mu)$ .

For  $\lambda$  a 4-th root of unity, consider the 8-dimensional representation  $W_\lambda$  of  $\mathfrak{A}_{3,1}(\mu)$  given by the following matrices

$$\begin{aligned} \rho_1(a) &= \begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & -\xi \mathbf{a} \end{pmatrix}, & \rho_1(b) &= \begin{pmatrix} \mathbf{0} & \mathbf{b}_{12} \\ \mathbf{b}_{21} & \mathbf{0} \end{pmatrix}, \\ \rho_1(x) &= \begin{pmatrix} \mathbf{0} & \mathbf{x} \\ id_4 & \mathbf{0} \end{pmatrix}, & \rho_1(y) &= \begin{pmatrix} \mathbf{y}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{y}_{22} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{a} &= \begin{pmatrix} \lambda & 0 & \mu(\lambda^3 - \lambda) & 0 \\ 0 & -\lambda & 0 & \mu(\lambda - \lambda^3) \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, & \mathbf{b}_{12} &= \begin{pmatrix} 0 & \xi \mu(\lambda^3 - \lambda) & 0 & \xi \mu^2(\lambda - \lambda^3) \\ 0 & 0 & 0 & 0 \\ 0 & 2\xi \lambda^3 & 0 & \xi \mu(\lambda - \lambda^3) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{b}_{21} &= \begin{pmatrix} 0 & -\lambda^3 & 0 & \mu(\lambda^3 - \lambda) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \mathbf{x} &= \begin{pmatrix} \mu(1 - \lambda^2) & 0 & \mu^2(\lambda^2 - 1) & 0 \\ 0 & \mu(1 - \lambda^2) & 0 & \mu^2(\lambda^2 - 1) \\ 2 & 0 & \mu(\lambda^2 - 1) & 0 \\ 0 & 2 & 0 & \mu(\lambda^2 - 1) \end{pmatrix}, \\ \mathbf{y}_{11} &= \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \mu^2(\lambda^2 - 1) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu(\lambda^2 - 1) \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \mathbf{y}_{22} &= \begin{pmatrix} 0 & \mu \lambda^2 & 0 & \frac{1}{2} \mu^2(1 - \lambda^2) \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \mu \\ 0 & 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

Assume

$$(6) \quad \sum_{0 \leq i, l \leq 1, 0 \leq j, k \leq 3} f_{i,j,k,l} x^i y^j a^k b^l = 0.$$

for some  $f_{i,j,k,l} \in \mathbb{k}$ . Applying this equation to the first vector of the canonical basis attached to the representation, we get that:

$$f_{i,j,0,0} + \lambda f_{i,j,1,0} + \lambda^2 f_{i,j,2,0} + \lambda^3 f_{i,j,3,0} = 0, \text{ for all } 0 \leq i \leq 1, 0 \leq j \leq 3.$$

Since this equation must hold for any  $\lambda$ , it follows that  $f_{i,j,k,0} = 0$  for all  $0 \leq i \leq 1$ , and  $0 \leq j, k \leq 3$ . To prove that the remaining coefficients are zero, we need another representation. For  $\lambda$  a 4-th root of unity, consider now the 16-dimensional representation  $U_\lambda$  given by

$$\begin{aligned} \rho_2(a) &= \begin{pmatrix} \mathbf{a} & \mathbf{0} & \mu(\mathbf{d} - \mathbf{a}) & \mathbf{0} & \mathbf{0} & \xi\mu(\mathbf{c} - \mathbf{b}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{a} & \mathbf{0} & \mu(\mathbf{a} - \mathbf{d}) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{a} & \mathbf{0} & \mathbf{0} & 2\xi\mathbf{c} & \mathbf{0} & -\xi\mu(\mathbf{b} + \mathbf{c}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{c} & \mathbf{0} & \mu\mathbf{c} & -\xi\mathbf{a} & \mathbf{0} & \xi\mu(\mathbf{a} - \mathbf{d}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi\mathbf{a} & \mathbf{0} & \xi\mu(\mathbf{d} - \mathbf{a}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c} & \mathbf{0} & \mathbf{0} & \xi\mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\xi\mathbf{a} \end{pmatrix}, \\ \rho_2(b) &= \begin{pmatrix} \mathbf{b} & \mathbf{0} & -\mu\mathbf{b} & \mathbf{0} & \mathbf{0} & \xi\mu(\mathbf{d} - \mathbf{a}) & \mathbf{0} & \xi\mu^2(\mathbf{a} - \mathbf{d}) \\ \mathbf{0} & -\mathbf{b} & \mathbf{0} & \mu\mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{b} & \mathbf{0} & \mathbf{0} & 2\xi\mathbf{d} & \mathbf{0} & \xi\mu(\mathbf{a} - \mathbf{d}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{d} & \mathbf{0} & \mu(\mathbf{d} - \mathbf{a}) & -\xi\mathbf{b} & \mathbf{0} & \xi\mu\mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi\mathbf{b} & \mathbf{0} & -\xi\mu\mathbf{b} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{d} & \mathbf{0} & \mathbf{0} & \xi\mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\xi\mathbf{b} \end{pmatrix}, \\ \rho_2(x) &= \begin{pmatrix} \mathbf{0}_8 & \mathbf{x} \\ id_8 & \mathbf{0}_8 \end{pmatrix}, \\ \rho_2(y) &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\mu^2}{2}(\mathbf{a}^2 - id_2) & \xi\mu\mathbf{ac} & \mathbf{0} & -\xi\mu^2\mathbf{ab} & \mathbf{0} \\ id_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi\mu\mathbf{ac} & \mathbf{0} & -\xi\mu^2\mathbf{ab} \\ \mathbf{0} & id_2 & \mathbf{0} & \mu(\mathbf{a}^2 - id_2) & \mathbf{0} & \mathbf{0} & \xi\mu\mathbf{ac} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & id_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi\mu\mathbf{ac} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mu\mathbf{a}^2 & \mathbf{0} & \frac{\mu^2}{2}(id_2 - \mathbf{a}^2) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -id_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -id_2 & \mathbf{0} & \mu id_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -id_2 & \mathbf{0} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{a} &= \begin{pmatrix} \lambda & \mathbf{0} \\ \mathbf{0} & -\xi\lambda \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{0} & \lambda^2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \lambda^3 & \mathbf{0} \\ \mathbf{0} & \xi\lambda^3 \end{pmatrix}, \quad \text{and} \\ \mathbf{x} &= \begin{pmatrix} \mu(id_2 - \mathbf{a}^2) & \mathbf{0} & \mu^2(\mathbf{a}^2 - id_2) & \mathbf{0} \\ \mathbf{0} & \mu(id_2 - \mathbf{a}^2) & \mathbf{0} & \mu^2(\mathbf{a}^2 - id_2) \\ 2id_2 & \mathbf{0} & \mu(\mathbf{a}^2 - id_2) & \mathbf{0} \\ \mathbf{0} & 2id_2 & \mathbf{0} & \mu(\mathbf{a}^2 - id_2) \end{pmatrix}. \end{aligned}$$

Applying the residual equation of (6) to the second vector of the canonical basis attached to the new representation, we get that

$$f_{i,j,0,1} + \lambda f_{i,j,1,1} + \lambda^2 f_{i,j,2,1} + \lambda^3 f_{i,j,3,1} = 0, \text{ for all } 0 \leq i \leq 1, 0 \leq j \leq 3$$

implying that  $f_{i,j,k,1} = 0$  for all  $0 \leq i \leq 1, 0 \leq j, k \leq 3$ . Therefore,  $B$  is a linearly independent set and  $\mathfrak{A}_{3,1}(\mu)$  is a lifting of  $\mathfrak{B}(V_{3,1}) \# \mathcal{K}$  for any  $\mu \in \mathbb{k}$ .  $\square$

**Proposition 5.10.** *Let  $H$  be a finite-dimensional Hopf algebra over  $\mathcal{K}$  such that its infinitesimal braiding is isomorphic to  $V_{3,1}$  or  $V_{3,3}$ . Then  $H \simeq \mathfrak{A}_{3,1}(\mu)$  or  $H \simeq \mathfrak{A}_{3,3}(\mu)$  for some  $\mu \in \mathbb{k}$ , respectively.*

*Proof.* Since  $H_{[0]} \simeq \mathcal{K}$ , we have that  $\text{gr } H = \mathfrak{B}(V) \# \mathcal{K}$  with  $V \simeq V_{3,1}$  or  $V \simeq V_{3,3}$ . Recall that  $\mathfrak{B}(V) \# \mathcal{K}$  is the algebra generated by  $x, y, a, b, c, d$  with  $x, y$  satisfying the relations of  $\mathfrak{B}(V)$ , see Proposition 4.8,  $a, b, c, d$  satisfying the relations of  $\mathcal{K}$ , and all together satisfying the relations giving the commutativity:

$$(7) \quad \begin{aligned} ax &= -\xi xa, & ay &= -ya - xc, & bx &= -\xi xb, & by &= -yb - xd, \\ cx &= \xi xc, & cy &= -yc + xa, & dx &= \xi xd, & dy &= -yd + xb. \end{aligned}$$

We prove the claim for  $V \simeq V_{3,1}$ . The proof for  $V \simeq V_{3,3}$  follows the same lines. In this case, we have that  $\Delta(x) = x \otimes 1 + d \otimes x + (\xi - 1)c \otimes y$  and  $\Delta(y) = y \otimes 1 + a \otimes y - \frac{\xi + 1}{2}b \otimes x$ , which implies

$$(8) \quad \Delta(x^2 - 2y^2) = (x^2 - 2y^2) \otimes 1 + a^2 \otimes (x^2 - 2y^2) \quad \text{and}$$

$$(9) \quad \Delta(xy + yx) = (xy + yx) \otimes 1 + 1 \otimes (xy + yx) - \xi ac \otimes (x^2 - 2y^2).$$

From the first equation, we get that the element  $x^2 - 2y^2 \in P_{1,a^2}((V_{3,1} \oplus \mathbb{k}) \# \mathcal{K}) = P_{1,a^2}(\mathcal{K}) = \mathbb{k}\{1 - a^2, ab\}$ . Then, there should exist  $\mu_1, \mu_2 \in \mathbb{k}$  such that

$$x^2 - 2y^2 = \mu_1(1 - a^2) + \mu_2 ab \quad \text{in } H.$$

However, a tedious calculation shows that (9) is possible only if  $\mu_2 = 0$ . In this case,

$$\begin{aligned} \Delta(xy + yx) &= (xy + yx) \otimes 1 + 1 \otimes (xy + yx) - \xi ac \otimes \mu_1(1 - a^2) \\ &= (xy + yx - \mu_1 \xi ac) \otimes 1 + 1 \otimes (xy + yx) + \xi \mu_1 ac \otimes a^2 \\ &= (xy + yx - \mu_1 \xi ac) \otimes 1 + 1 \otimes (xy + yx - \mu_1 \xi ac) + \xi \mu_1 ac \otimes a^2 + 1 \otimes \mu_1 \xi ac \\ &= (xy + yx - \mu_1 \xi ac) \otimes 1 + 1 \otimes (xy + yx - \mu_1 \xi ac) + \Delta(\xi \mu_1 ac), \end{aligned}$$

which implies that  $xy + yx - \mu_1 \xi ac$  is a primitive element in  $H$ . Thus, we must have that

$$xy + yx = \xi \mu_1 ac.$$

Finally, for  $y^4$  we have

$$\begin{aligned} \Delta(y^4) &= y^4 \otimes 1 + 1 \otimes y^4 - \mu_1 y^2 a^2 \otimes (1 - a^2) + \mu_1(1 - a^2) \otimes y^2 + \mu_1 xb \otimes ya^2 + \mu_1 xc \otimes y \\ &\quad + \frac{1 + \xi}{2} \mu_1 xa \otimes xa^2 + \frac{1 - \xi}{2} \mu_1^2 ac \otimes ab + \frac{\xi - 1}{2} \mu_1^2 ab \otimes ac - \frac{1 + \xi}{2} \mu_1 xd \otimes x \\ &\quad + \frac{1}{2} \mu_1^2 (1 - a^2) \otimes (1 - a^2) - \xi \mu_1 ab \otimes xy - \xi \mu_1 ac \otimes xya^2 \end{aligned}$$

which implies that  $y^4 = -\mu_1 y^2(1 - a^2) - \frac{1}{2} \mu_1^2 (1 - a^2)$ .  $\square$

We end the paper with the classification of finite-dimensional Hopf algebras over  $\mathcal{K}$  such that their infinitesimal braiding is an indecomposable module in  ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$ .

**Proof of Theorem B.** Since  $H_{[0]} \simeq \mathcal{K}$ , by Theorem 5.1 we have that  $\text{gr } H = R \# \mathcal{K}$  and  $R \simeq \mathfrak{B}(V)$  with  $V$  a simple object in  ${}_{\mathcal{K}}^{\mathcal{K}}\mathcal{YD}$ . If  $V \simeq \mathbb{k}_{\chi}, \mathbb{k}_{\chi^3}, V_{2,1}$  or  $V_{2,3}$ , then  $H \simeq \mathfrak{B}(V) \# \mathcal{K}$  by Propositions 5.2 and 5.3. If  $V \simeq V_{3,1}$  or  $V \simeq V_{3,3}$ , then by Proposition 5.10 it follows that  $H \simeq \mathfrak{A}_{3,1}(\mu)$  or  $H \simeq \mathfrak{A}_{3,3}(\mu)$  for some  $\mu \in \mathbb{k}$ , respectively.

Conversely, it is clear that the algebras listed in items (i) and (iii) are liftings of Nichols algebras over  $\mathcal{K}$ . The Hopf algebras  $\mathfrak{A}_{3,1}(\mu)$  and  $\mathfrak{A}_{3,3}(\mu)$  are liftings of Nichols algebras over  $\mathcal{K}$  for all  $\mu \in \mathbb{k}$  by Lemma 5.9.

Finally, two algebras from different families are not isomorphic as Hopf algebras since their infinitesimal braidings are not isomorphic as Yetter–Drinfeld modules  $\square$

## 6. APPENDIX

We are interested to give a full proof of Remark 3.5. We perform a case by case analysis.

Let  $V_{i,j} \in {}^{\mathcal{K}}\mathcal{YD}$  with  $(i,j) \in \Lambda$  be a simple module. Throughout the section, we suppose the braiding is of diagonal type, *i.e.* there exists a basis  $\{v_1, v_2\}$  of  $V_{i,j}$  such that  $c(v_i \otimes v_j) = q_{ij}v_j \otimes v_i$ ,  $1 \leq i, j \leq 2$ . We will see that in each case we arrive at a contradiction. If we write  $v_i = \alpha_{i1}e_1 + \alpha_{i2}e_2$ ,  $i = 1, 2$ , we must have

$$\begin{aligned} c(v_i \otimes v_j) &= c((\alpha_{i1}e_1 + \alpha_{i2}e_2) \otimes (\alpha_{j1}e_1 + \alpha_{j2}e_2)) \\ &= \alpha_{i1}\alpha_{j1}c(e_1 \otimes e_1) + \alpha_{i1}\alpha_{j2}c(e_1 \otimes e_2) + \alpha_{i2}\alpha_{j1}c(e_2 \otimes e_1) + \alpha_{i2}\alpha_{j2}c(e_2 \otimes e_2), \end{aligned}$$

but on the other hand,

$$\begin{aligned} c(v_i \otimes v_j) &= q_{ij}v_j \otimes v_i = q_{ij}(\alpha_{j1}e_1 + \alpha_{j2}e_2) \otimes (\alpha_{i1}e_1 + \alpha_{i2}e_2) \\ &= q_{ij}\alpha_{i1}\alpha_{j1}e_1 \otimes e_1 + q_{ij}\alpha_{i2}\alpha_{j1}e_1 \otimes e_2 + q_{ij}\alpha_{i1}\alpha_{j2}e_2 \otimes e_1 + q_{ij}\alpha_{i2}\alpha_{j2}e_2 \otimes e_2. \end{aligned}$$

- **Case  $j = 0$  and  $i \in \{1, 3\}$ :** By Proposition 3.4, the braiding is given by

$$c\left(\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix}\right) \otimes \begin{Bmatrix} e_1 & e_2 \end{Bmatrix} = \begin{Bmatrix} e_1 \otimes e_1 & e_2 \otimes e_1 + 2e_1 \otimes e_2 \\ -e_1 \otimes e_2 & e_2 \otimes e_2 \end{Bmatrix}.$$

Then we must have that

$$\begin{aligned} (10) \quad & q_{12}\alpha_{11}\alpha_{22} = \alpha_{11}\alpha_{22} \\ (11) \quad & q_{12}\alpha_{12}\alpha_{21} = 2\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \\ (12) \quad & q_{21}\alpha_{12}\alpha_{21} = \alpha_{12}\alpha_{21} \\ (13) \quad & q_{21}\alpha_{11}\alpha_{22} = 2\alpha_{12}\alpha_{21} - \alpha_{22}\alpha_{11}. \end{aligned}$$

If we denote  $\det := \det \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} \neq 0$ , from (10) and (11) we get that  $q_{12}\det = -\det$ .

This implies that  $q_{12} = -1$  and whence  $\alpha_{11}\alpha_{22} = 0$ . Similarly from (13) and (12) we get that  $q_{21}\det = -\det$ ,  $q_{21} = -1$  and  $\alpha_{12}\alpha_{21} = 0$ , which implies that  $\det = 0$ , a contradiction.

- **Case  $j = 2$  and  $i \in \{0, 2\}$ :** By Proposition 3.4, the braiding is given by

$$c\left(\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix}\right) \otimes \begin{Bmatrix} e_1 & e_2 \end{Bmatrix} = \begin{Bmatrix} e_1 \otimes e_1 & -e_2 \otimes e_1 + 2e_1 \otimes e_2 \\ e_1 \otimes e_2 & e_2 \otimes e_2 \end{Bmatrix}.$$

Then we have

$$\begin{aligned} (14) \quad & q_{12}\alpha_{11}\alpha_{22} = -\alpha_{11}\alpha_{22} \\ (15) \quad & q_{12}\alpha_{12}\alpha_{21} = 2\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} \\ (16) \quad & q_{21}\alpha_{12}\alpha_{21} = -\alpha_{12}\alpha_{21} \\ (17) \quad & q_{21}\alpha_{11}\alpha_{22} = 2\alpha_{12}\alpha_{21} + \alpha_{22}\alpha_{11}. \end{aligned}$$

Computing (14)-(15)+(17)-(16) yields  $(q_{12} + q_{21})\det = -2\det$ , which implies that  $q_{21} = -q_{12} - 2$ . Analogously, (14)+(17) yields  $\alpha_{12}\alpha_{21} = -\alpha_{11}\alpha_{22}$ . On the other hand,  $q_{11}\alpha_{11}^2 = \alpha_{11}^2$  and  $q_{11}\alpha_{12}^2 = \alpha_{12}^2$ , which implies that  $q_{11} = 1$ , since  $\alpha_{12} \neq 0$  or  $\alpha_{11} \neq 0$ . As  $q_{11}\alpha_{11}\alpha_{12} = -\alpha_{11}\alpha_{12}$ , we get that  $\alpha_{11}\alpha_{12} = 0$ . Similarly, using the equations of  $q_{22}$ , it follows that  $\alpha_{21}\alpha_{22} = 0$ . Note that  $\alpha_{11}\alpha_{12} = 0$  and  $\alpha_{21}\alpha_{22} = 0$  imply that both columns of  $(\alpha_{ij})_{1 \leq i, j \leq 2}$  have zero elements. Therefore, this matrix is a diagonal matrix because it is a non-singular matrix. But  $\alpha_{12}\alpha_{21} = -\alpha_{11}\alpha_{22}$ , implying this matrix has determinant zero.

- **Case  $j = 1$  and  $i$  arbitrary:** By Proposition 3.4, the braiding is given by

$$c\left(\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix}\right) \otimes \begin{Bmatrix} e_1 & e_2 \end{Bmatrix} = \begin{Bmatrix} \lambda_1^3 e_1 \otimes e_1 & \xi \lambda_1^3 e_2 \otimes e_1 + (\lambda_1^3 - \xi \lambda_1) e_1 \otimes e_2 \\ \lambda_1 e_1 \otimes e_2 & -\xi \lambda_1 e_2 \otimes e_2 + \frac{1}{2}(\lambda_1^3 + \xi \lambda_1) e_1 \otimes e_1 \end{Bmatrix}.$$

(i) Assume  $i = 2$ , then

$$(18) \quad q_{ij}\alpha_{i1}\alpha_{j1} = -\alpha_{i1}\alpha_{j1} - \frac{1+\xi}{2}\alpha_{i2}\alpha_{j2}$$

$$(19) \quad q_{ij}\alpha_{i2}\alpha_{j2} = \xi\alpha_{i2}\alpha_{j2}$$

$$(20) \quad q_{ij}\alpha_{i1}\alpha_{j2} = -\xi\alpha_{i1}\alpha_{j2}$$

$$(21) \quad q_{ij}\alpha_{i2}\alpha_{j1} = (\xi - 1)\alpha_{i1}\alpha_{j2} - \alpha_{i2}\alpha_{j1}.$$

Since in this case the eigenvalues of  $\tilde{c} := \tau \circ c$  are  $-1$  and  $\pm\xi$ , we have that  $q_{11} \in \{-1, \pm\xi\}$ . Suppose  $q_{11} = -1$ . Then, by (19) we get  $-\alpha_{12}^2 = \xi\alpha_{12}^2$  and hence  $\alpha_{12} = 0$ . This implies that  $\alpha_{11}\alpha_{22} = \det \neq 0$ . But from (21) it follows that  $q_{12}\alpha_{12}\alpha_{21} = (\xi - 1)\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$ , which implies that  $\det = \alpha_{11}\alpha_{22} = 0$ , a contradiction. Assume now that  $q_{11} = -\xi$ . Then, by (19) we get  $-\xi\alpha_{12}^2 = \xi\alpha_{12}^2$  which implies that  $\alpha_{12} = 0$  and  $\alpha_{11}\alpha_{22} \neq 0$ . But, again by (21) we obtain  $q_{12}\alpha_{12}\alpha_{21} = (\xi 1)\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$  from which follows that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction. Finally, suppose that  $q_{11} = \xi$ . Then, by (18) we have that  $\xi\alpha_{11}^2 = -\alpha_{11}^2 - \frac{1+\xi}{2}\alpha_{12}^2$  and consequently  $\alpha_{12}^2 = -2\alpha_{11}^2$ . On the other hand, (20) yields that  $\xi\alpha_{11}\alpha_{12} = -\xi\alpha_{11}\alpha_{12}$  and hence  $\alpha_{11}\alpha_{12} = 0$ , which implies that  $\alpha_{11} = \alpha_{12} = 0$ , a contradiction.

(ii) Assume  $i = 3$ , then

$$(22) \quad q_{ij}\alpha_{i1}\alpha_{j1} = \xi\alpha_{i1}\alpha_{j1} + \frac{1+\xi}{2}\alpha_{i2}\alpha_{j2}$$

$$(23) \quad q_{ij}\alpha_{i2}\alpha_{j2} = -\alpha_{i2}\alpha_{j2}$$

$$(24) \quad q_{ij}\alpha_{i1}\alpha_{j2} = -\alpha_{i1}\alpha_{j2}$$

$$(25) \quad q_{ij}\alpha_{i2}\alpha_{j1} = (\xi - 1)\alpha_{i1}\alpha_{j2} - \xi\alpha_{i2}\alpha_{j1}.$$

In this case, the eigenvalues of  $\tilde{c}$  are again  $-1$  and  $\pm\xi$ . Then  $q_{11} \in \{-1, \pm\xi\}$ . Suppose first that  $q_{11} = -1$ . Then, by (22) we have that  $-\alpha_{11}^2 = \xi\alpha_{11}^2 + \frac{1+\xi}{2}\alpha_{12}^2$  which implies that  $\alpha_{12}^2 = -2\alpha_{11}^2$ . Also, (23) gives  $q_{12}\alpha_{12}\alpha_{22} = -\alpha_{12}\alpha_{22}$  and  $q_{21}\alpha_{12}\alpha_{22} = -\alpha_{12}\alpha_{22}$ . Thus, if  $\alpha_{12}\alpha_{22} \neq 0$  we would have that  $-1 = q_{11} = q_{12} = q_{21}$ , which is impossible. If  $\alpha_{12} = 0$ , then  $\alpha_{11} = 0$ , a contradiction. Hence, we should have that  $\alpha_{22} = 0$ . But in such a case, (25) yields  $q_{21}\alpha_{11}\alpha_{22} = (\xi - 1)\alpha_{12}\alpha_{21} - \xi\alpha_{11}\alpha_{22}$  from which follows that  $\alpha_{12}\alpha_{21} = 0$ , a contradiction. Suppose  $q_{11} = -\xi$ , then by (23) we have that  $-\xi\alpha_{12}^2 = -\alpha_{12}^2$  and whence  $\alpha_{12} = 0$ . Also, from (22) it follows that  $-\xi\alpha_{11}^2 = \xi\alpha_{11}^2 + \frac{1+\xi}{2}\alpha_{12}^2$  which implies  $\alpha_{11} = 0$ , a contradiction. Assume finally that  $q_{11} = \xi$ , then, by (23) we have that  $\xi\alpha_{12}^2 = -\alpha_{12}^2$  and then  $\alpha_{12} = 0$ . Besides, by (25) we have that  $q_{12}\alpha_{12}\alpha_{21} = (\xi - 1)\alpha_{11}\alpha_{22} - \xi\alpha_{12}\alpha_{21}$  which implies that  $\alpha_{11}\alpha_{12} = 0$ , a contradiction.

(iii) Assume  $i = 0$ , then

$$(26) \quad q_{ij}\alpha_{i1}\alpha_{j1} = \alpha_{i1}\alpha_{j1} + \frac{1+\xi}{2}\alpha_{i2}\alpha_{j2}$$

$$(27) \quad q_{ij}\alpha_{i2}\alpha_{j2} = -\xi\alpha_{i2}\alpha_{j2}$$

$$(28) \quad q_{ij}\alpha_{i1}\alpha_{j2} = \xi\alpha_{i1}\alpha_{j2}$$

$$(29) \quad q_{ij}\alpha_{i2}\alpha_{j1} = (1 - \xi)\alpha_{i1}\alpha_{j2} + \alpha_{i2}\alpha_{j1}.$$

Here, the eigenvalues of  $\tilde{c}$  are  $1$  and  $\pm\xi$ . Then  $q_{11} \in \{1, \pm\xi\}$ . Suppose  $q_{11} = 1$ , then by (27) we have that  $\alpha_{12}^2 = -\xi\alpha_{12}^2$  and  $\alpha_{12} = 0$ . Also, by (29) we have that  $q_{12}\alpha_{12}\alpha_{21} = (1 - \xi)\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}$  from which follows that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction. Suppose now that  $q_{11} = -\xi$ . Then, by (26) we obtain  $-\xi\alpha_{11}^2 = \alpha_{11}^2 + \frac{1+\xi}{2}\alpha_{12}^2$  and whence  $\alpha_{12}^2 = -2\alpha_{11}^2$ . Also, by (28) we have that  $-\xi\alpha_{11}\alpha_{12} = \xi\alpha_{11}\alpha_{12}$  and consequently  $\alpha_{11}\alpha_{12} = 0$ , which implies  $\alpha_{11} = \alpha_{12} = 0$ , a contradiction. Assume  $q_{11} = \xi$ , then by (27) we have  $\xi\alpha_{12}^2 = -\xi\alpha_{12}^2$ , implying that  $\alpha_{12} = 0$ . Also, (29) yields that  $q_{12}\alpha_{12}\alpha_{21} = (1 - \xi)\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}$  from which follows that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction.

(iv) Assume  $i = 1$ , then

$$(30) \quad q_{ij}\alpha_{i1}\alpha_{j1} = -\xi\alpha_{i1}\alpha_{j1} - \frac{1+\xi}{2}\alpha_{i2}\alpha_{j2}$$

$$(31) \quad q_{ij}\alpha_{i2}\alpha_{j2} = \alpha_{i2}\alpha_{j2}$$

$$(32) \quad q_{ij}\alpha_{i1}\alpha_{j2} = \alpha_{i1}\alpha_{j2}$$

$$(33) \quad q_{ij}\alpha_{i2}\alpha_{j1} = (1-\xi)\alpha_{i1}\alpha_{j2} + \xi\alpha_{i2}\alpha_{j1}.$$

In this case, the eigenvalues of  $\tilde{c}$  are 1 and  $\pm\xi$ , then  $q_{11} \in \{1, \pm\xi\}$ . Suppose  $q_{11} = 1$ , then (30) gives  $\alpha_{11}^2 = -\xi\alpha_{11}^2 - \frac{1+\xi}{2}\alpha_{12}^2$  which implies that  $\alpha_{12}^2 = -2\alpha_{11}^2$ . Also, (31) gives  $q_{12}\alpha_{12}\alpha_{22} = \alpha_{12}\alpha_{22}$  and (31) gives  $q_{21}\alpha_{12}\alpha_{22} = \alpha_{12}\alpha_{22}$ . If  $\alpha_{12}\alpha_{22} \neq 0$ , then  $1 = q_{11} = q_{12} = q_{21}$ , which is impossible. If  $\alpha_{12} = 0$ , then also  $\alpha_{11} = 0$ , a contradiction. Thus,  $\alpha_{22} = 0$ . In such a case, by (33) we have  $q_{21}\alpha_{11}\alpha_{22} = (1-\xi)\alpha_{12}\alpha_{21} + \xi\alpha_{11}\alpha_{22}$  which implies that  $\alpha_{12}\alpha_{21} = 0$ , a contradiction. Suppose now  $q_{11} = -\xi$ , then by (31) we have  $-\xi\alpha_{12}^2 = \alpha_{12}^2$  and  $\alpha_{12} = 0$ . Also, (33) yields that  $q_{12}\alpha_{12}\alpha_{21} = (1-\xi)\alpha_{11}\alpha_{22} + \xi\alpha_{12}\alpha_{21}$  which implies that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction. Assume finally that  $q_{11} = \xi$ , then by (31) we have that  $\xi\alpha_{12}^2 = \alpha_{12}^2$  and whence  $\alpha_{12} = 0$ . Besides, (33) gives  $q_{12}\alpha_{12}\alpha_{21} = (1-\xi)\alpha_{11}\alpha_{22} + \xi\alpha_{12}\alpha_{21}$  which implies that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction.

Finally, we prove the last case, also by a case-by-case argument.

• **Case  $j = 3$  and  $i$  arbitrary:** By Proposition 3.4, the braiding is given by

$$c\left(\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix}\right) \otimes \begin{Bmatrix} e_1 & e_2 \end{Bmatrix} = \begin{Bmatrix} \lambda_1 e_1 \otimes e_1 & -\xi\lambda_1 e_2 \otimes e_1 + (\lambda_1 + \xi\lambda_1^3)e_1 \otimes e_2 \\ \lambda_1^3 e_1 \otimes e_2 & \xi\lambda_1^3 e_2 \otimes e_2 + \frac{1}{2}(\lambda_1 - \xi\lambda_1^3)e_1 \otimes e_1 \end{Bmatrix}.$$

(i) Assume  $i = 2$ , then

$$(34) \quad q_{ij}\alpha_{i1}\alpha_{j1} = -\alpha_{i1}\alpha_{j1} + \frac{\xi-1}{2}\alpha_{i2}\alpha_{j2}$$

$$(35) \quad q_{ij}\alpha_{i2}\alpha_{j2} = -\xi\alpha_{i2}\alpha_{j2}$$

$$(36) \quad q_{ij}\alpha_{i1}\alpha_{j2} = \xi\alpha_{i1}\alpha_{j2}$$

$$(37) \quad q_{ij}\alpha_{i2}\alpha_{j1} = -(\xi+1)\alpha_{i1}\alpha_{j2} - \alpha_{i2}\alpha_{j1}.$$

Since the eigenvalues of  $\tilde{c}$  are  $-1$  and  $\pm\xi$ , we have that  $q_{11} \in \{-1, \pm\xi\}$ . Suppose  $q_{11} = -1$ , then by (35) it follows that  $-\alpha_{12}^2 = -\xi\alpha_{12}^2$  which implies that  $\alpha_{12} = 0$ . Also, (37) gives that  $q_{12}\alpha_{12}\alpha_{21} = -(\xi+1)\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$  from which follows that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction.

Suppose now that  $q_{11} = -\xi$ , then by (34) we obtain  $-\xi\alpha_{11}^2 = -\alpha_{11}^2 + \frac{\xi-1}{2}\alpha_{12}^2$  and therefore  $\alpha_{12}^2 = -2\alpha_{11}^2$ . Also, by (36) we have that  $-\xi\alpha_{11}\alpha_{12} = \xi\alpha_{11}\alpha_{12}$  which implies that  $\alpha_{11}\alpha_{12} = 0$  and hence  $\alpha_{11} = \alpha_{12} = 0$ , a contradiction. Suppose then that  $q_{11} = \xi$ . In this case, (35) yields  $\xi\alpha_{12}^2 = -\xi\alpha_{12}^2$  which implies that  $\alpha_{12} = 0$ . But (37) gives  $q_{12}\alpha_{12}\alpha_{21} = -(\xi+1)\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$  from which follows that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction.

(ii) Assume  $i = 3$ , then

$$(38) \quad q_{ij}\alpha_{i1}\alpha_{j1} = -\xi\alpha_{i1}\alpha_{j1} + \frac{1-\xi}{2}\alpha_{i2}\alpha_{j2}$$

$$(39) \quad q_{ij}\alpha_{i2}\alpha_{j2} = -\alpha_{i2}\alpha_{j2}$$

$$(40) \quad q_{ij}\alpha_{i1}\alpha_{j2} = -\alpha_{i1}\alpha_{j2}$$

$$(41) \quad q_{ij}\alpha_{i2}\alpha_{j1} = -(1+\xi)\alpha_{i1}\alpha_{j2} + \xi\alpha_{i2}\alpha_{j1}.$$

Note that the eigenvalues of  $\tilde{c}$  are  $-1$  and  $\pm\xi$ ; then  $q_{11} \in \{-1, \pm\xi\}$ . If  $q_{11} = -1$ , then by (38) we have that  $-\alpha_{11}^2 = -\xi\alpha_{11}^2 + \frac{1-\xi}{2}\alpha_{12}^2$ , which implies that  $\alpha_{12}^2 = -2\alpha_{11}^2$ . Also, by (39) we have that  $q_{12}\alpha_{12}\alpha_{22} = -\alpha_{12}\alpha_{22}$   $q_{21}\alpha_{12}\alpha_{22} = -\alpha_{12}\alpha_{22}$ . If  $\alpha_{12}\alpha_{22} \neq 0$ , we would have that  $q_{11} = q_{12} = q_{21}$ , a contradiction. If  $\alpha_{12} = 0$ , then  $\alpha_{11} = 0$ , also a contradiction. Thus,  $\alpha_{22} = 0$ . In such a case, (41) yields  $q_{21}\alpha_{11}\alpha_{22} = -(1+\xi)\alpha_{12}\alpha_{21} + \xi\alpha_{22}\alpha_{11}$  from which follows that  $\alpha_{12}\alpha_{21} = 0$ , a contradiction. If  $q_{11} = -\xi$ , then by (39) we have that  $-\xi\alpha_{12}^2 =$



$-\alpha_{12}^2$  giving  $\alpha_{12} = 0$ . Also, (41) yields  $q_{12}\alpha_{12}\alpha_{21} = -(1+\xi)\alpha_{11}\alpha_{22} + \xi\alpha_{12}\alpha_{21}$  which implies that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction. If  $q_{11} = \xi$ , we have by (39) that  $\xi\alpha_{12}^2 = -\alpha_{12}^2$  which implies  $\alpha_{12} = 0$ . Also, by (41) we have that  $q_{12}\alpha_{12}\alpha_{21} = -(1+\xi)\alpha_{11}\alpha_{22} + \xi\alpha_{12}\alpha_{21}$  which yields  $\alpha_{11}\alpha_{22} = 0$ , a contradiction.

(iii) Assume  $i = 0$ , then

$$(42) \quad q_{ij}\alpha_{i1}\alpha_{j1} = \alpha_{i1}\alpha_{j1} + \frac{1-\xi}{2}\alpha_{i2}\alpha_{j2}$$

$$(43) \quad q_{ij}\alpha_{i2}\alpha_{j2} = \xi\alpha_{i2}\alpha_{j2}$$

$$(44) \quad q_{ij}\alpha_{i1}\alpha_{j2} = -\xi\alpha_{i1}\alpha_{j2}$$

$$(45) \quad q_{ij}\alpha_{i2}\alpha_{j1} = (1+\xi)\alpha_{i1}\alpha_{j2} + \alpha_{i2}\alpha_{j1}.$$

Since the eigenvalues of  $\tilde{c}$  are 1 and  $\pm\xi$ , then  $q_{11} \in \{1, \pm\xi\}$ . If  $q_{11} = 1$ , then by (43) we have that  $\alpha_{12} = 0$ , and by (45) it follows that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction. If  $q_{11} = -\xi$ , then by (43) we have that  $\alpha_{12} = 0$ , and by (45) we have that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction too. Finally, if  $q_{11} = \xi$ , then, by (42) we have that  $\alpha_{12}^2 = -2\alpha_{11}^2$  and by (44)  $\alpha_{11}\alpha_{12} = 0$ . This implies that  $\alpha_{11} = \alpha_{12} = 0$ , a contradiction.

(iv) Assume  $i = 1$ , then

$$(46) \quad q_{ij}\alpha_{i1}\alpha_{j1} = \xi\alpha_{i1}\alpha_{j1} + \frac{\xi-1}{2}\alpha_{i2}\alpha_{j2}$$

$$(47) \quad q_{ij}\alpha_{i2}\alpha_{j2} = \alpha_{i2}\alpha_{j2}$$

$$(48) \quad q_{ij}\alpha_{i1}\alpha_{j2} = \alpha_{i1}\alpha_{j2}$$

$$(49) \quad q_{ij}\alpha_{i2}\alpha_{j1} = (1+\xi)\alpha_{i1}\alpha_{j2} - \xi\alpha_{i2}\alpha_{j1}.$$

Since the eigenvalues of  $\tilde{c}$  are 1 and  $\pm\xi$ , we have that  $q_{11} \in \{1, \pm\xi\}$ . if  $q_{11} = 1$ , then by (46) we have that  $\alpha_{12}^2 = -2\alpha_{11}^2$ . Also, from (47) and (47) it follows that  $q_{12}\alpha_{12}\alpha_{22} = \alpha_{12}\alpha_{22}$  and  $q_{21}\alpha_{12}\alpha_{22} = \alpha_{12}\alpha_{22}$ . If  $\alpha_{12}\alpha_{22} \neq 0$ , we would have that  $1 = q_{11} = q_{12} = q_{21}$ , a contradiction. If  $\alpha_{12} = 0$ , then  $\alpha_{11} = 0$ , a contradiction too. Thus, we should have that  $\alpha_{22} = 0$ . But in such a case, (49) yields that  $\alpha_{21}\alpha_{12} = 0$ , which is impossible. If  $q_{11} = -\xi$ , then (47) implies that  $\alpha_{12} = 0$  and (49) implies that  $\alpha_{11}\alpha_{22} = 0$ , a contradiction. Finally, if  $q_{11} = \xi$ , then (47) implies  $\alpha_{12} = 0$  and (49) implies  $\alpha_{11}\alpha_{22} = 0$ , which is impossible.  $\square$

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